

# A RANDOM SCHRÖDINGER OPERATOR ASSOCIATED WITH THE VERTEX REINFORCED JUMP PROCESS ON INFINITE GRAPHS

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**ABSTRACT.** This paper concerns the Vertex reinforced jump process (VRJP), the Edge reinforced random walk (ERRW) and their link with a random Schrödinger operator. On infinite graphs, we define a 1-dependent random potential  $\beta$  extending that defined in [18] on finite graphs, and consider its associated random Schrödinger operator  $H_\beta$ . We construct a random function  $\psi$  as a limit of martingales, such that  $\psi = 0$  when the VRJP is recurrent, and  $\psi$  is a positive generalized eigenfunction of the random Schrödinger operator with eigenvalue 0, when the VRJP is transient. Then we prove a representation of the VRJP on infinite graphs as a mixture of Markov jump processes involving the function  $\psi$ , the Green function of the random Schrödinger operator and an independent Gamma random variable. On  $\mathbb{Z}^d$ , we deduce from this representation a zero-one law for recurrence or transience of the VRJP and the ERRW, and a functional central limit theorem for the VRJP and the ERRW at weak reinforcement in dimension  $d \geq 3$ , using estimates of [10, 8]. Finally, we deduce recurrence of the ERRW in dimension  $d = 2$  for any initial constant weights (using the estimates of Merkl and Rolles, [14, 16]), thus giving a full answer to the old question of Diaconis. We also raise some questions on the links between recurrence/transience of the VRJP and localization/delocalization of the random Schrödinger operator  $H_\beta$ .

## 1. INTRODUCTION

This paper concerns the Vertex Reinforced Jump Process (VRJP) and the Edge Reinforced Random Walk (ERRW) and their relation with a random Schrödinger operator associated with a stationary 1-dependent random potential (i.e. the potential is independent at distance larger or equal to 2).

The VRJP is a continuous time self-interacting process introduced in [5], investigated on trees in [3, 2] and on general graphs in [18, 19]. We first recall its definition. Let  $\mathcal{G} = (V, E)$  be a non-directed graph with finite degree at each vertex. We write  $i \sim j$  if  $i \in V$ ,  $j \in V$  and  $\{i, j\}$  is an edge of the graph. We always assume that the graph is connected and has no trivial loops (i.e. vertex  $i$  such that  $i \sim i$ ). Let  $(W_{i,j})_{i \sim j}$  be a set of positive conductances,  $W_{i,j} > 0$ ,  $W_{i,j} = W_{j,i}$ . The VRJP is the continuous-time process  $(Y_s)_{s \geq 0}$  on  $V$ , starting at time 0 at some vertex  $i_0 \in V$ , which, conditionally on the past at time  $s$ , if  $Y_s = i$ , jumps to a neighbour  $j$  of  $i$  at rate

$$W_{i,j} L_j(s),$$

where

$$L_j(s) := 1 + \int_0^s \mathbb{1}_{\{Y_u = j\}} du.$$

In [18], Sabot and Tarrès introduced the following time change of the VRJP

$$Z_t = Y_{D^{-1}(t)},$$

where  $D(s)$  is the following increasing function

$$(1.1) \quad D(s) = \sum_{i \in V} (L_i^2(s) - 1).$$

Denote  $\mathbb{P}_{i_0}^{\text{VRJP}}$  the law of  $(Z_t)$  starting from the vertex  $i_0$ . When the graph is finite it is proved in [18] Theorem 2 that the time-changed VRJP  $Z$  is a mixture of Markov jump processes. More precisely, there exists a random field  $(u_j)_{j \in V}$  such that  $Z$  is a mixture of Markov jump processes with jump rates from  $i$  to  $j$

$$\frac{1}{2} W_{i,j} e^{u_j - u_i}.$$

The law of the field  $(u_j)$  is explicit, cf [18] Theorem 2 and forthcoming Theorem B. It appears to be a marginal of a supersymmetric sigma-field which had been investigated previously by Disertori, Spencer, Zirnbauer (cf [9], [10], [22]). As a consequence of this representation and of [9], [10], it was proved in [18] the following : when the graph has bounded degree, there exists a  $0 < \lambda_0$  such that if  $W_{i,j} \leq \lambda_0$  then the VRJP is positively recurrent, more precisely,  $Z$  is a mixture of positive recurrent Markov Jump processes. When the graph is the grid  $\mathbb{Z}^d$ , with  $d \geq 3$ , there exists  $\lambda_1 < +\infty$  such that if  $W_{i,j} \geq \lambda_1$ , the VRJP is transient. Hence, it shows a phase transition between recurrence and transience in dimension  $d \geq 3$ . The question of the representation of the VRJP on infinite graphs as a mixture of Markov jump processes is non trivial, especially in the transient case. It is possible to prove such a representation by a weak convergence argument, following [15], but it gives few information on the mixing law. In this paper we prove such a representation involving the Green function and a generalized eigenfunction of a random Schrödinger operator.

Let us give a flavor of the main results of the paper in the case of the VRJP on  $\mathbb{Z}^d$  with  $W_{i,j} = W$  constant. We construct a positive 1-dependent random potential  $(\beta_j)_{j \in \mathbb{Z}^d}$  (i.e. two subset of the  $\beta$ 's are independent if their indices are at least at distance 2) and with marginal given by inverse of Inverse Gaussian law with parameters  $1/(dW)$ . This field is a natural extension to infinite graphs of the field defined by Sabot, Tarrès, Zeng in [20]. We consider the random Schrödinger operator

$$H_\beta = -W\Delta + V,$$

where  $\Delta$  is the usual discrete (non-positive) Laplacian and  $V$  is the multiplication operator by  $V_j = 2\beta_j - 2dW$ . Hence, it corresponds to the Anderson model with a random potential which is not i.i.d. but only stationary and 1-dependent. When the VRJP is transient we prove that there exists a positive generalized eigenfunction  $\psi$  of  $H_\beta$  with eigenvalue 0, stationary and ergodic. Let  $(G(i, j))_{i \in \mathbb{Z}^d, j \in \mathbb{Z}^d}$  be defined by

$$G(i, j) = \hat{G}(i, j) + \frac{1}{2} \gamma^{-1} \psi(i) \psi(j),$$

where  $\hat{G} = (H_\beta)^{-1}$  is the Green function (which happens to be well-defined) and  $\gamma$  is an extra random variable independent of the field  $\beta$  with law  $\text{Gamma}(\frac{1}{2})$ . We prove the following representation for the VRJP : the time-changed VRJP  $Z$  starting from the point  $i_0$  is a mixture of Markov jump processes with jump rates from  $i$  to  $j$

$$(1.2) \quad \frac{1}{2} W_{i,j} \frac{G(i_0, j)}{G(i_0, i)}$$

When the VRJP is recurrent the same representation is valid with  $\psi = 0$ . In fact, the function  $\psi$  is the a.s. limit of a martingale, the limit being positive when the VRJP is transient and 0 when the VRJP is recurrent. It is remarkable that when the VRJP is recurrent it can be represented as a mixture with  $\beta$ -measurable jump rates, but when the VRJP is transient it involves an extra independent Gamma random variable. This representation extends to infinite graphs the representation given in [20] for finite graphs. An interesting feature appears in the transient case, where the generalized eigenfunction  $\psi$  is involved in the representation. We suspect that recurrence/transience of the VRJP is related to localization/delocalization of the random Schrödinger operator  $H_\beta$  at the bottom of the spectrum.

The representation (1.2) has several consequences on the VRJP and the ERRW. The ERRW is a reinforced process introduced by Diaconis and Coppersmith in 86 (see Section 2.4 for a definition). A famous open question raised by Diaconis, is that of the recurrence of the 2-dimensional ERRW, see [4, 17, 11, 16] for early references. Important progress have been done recently in the understanding of this process. In particular, in [18], an explicit relation between the ERRW and the VRJP was stated, thus somehow reducing the analysis of the ERRW to that of the VRJP. In [18, 1], it was proved by rather different methods that the ERRW on any graph with bounded degree at strong enough reinforcement is positive recurrent. In [8], it was proved that the ERRW is transient on  $\mathbb{Z}^d$ ,  $d \geq 3$ , at weak reinforcement.

The representation (1.2) allows to complete the picture both in dimension 2 and in the transient regime. More precisely, we prove a functional central limit theorem for the ERRW and for the discrete time process associated with the VRJP in dimension  $d \geq 3$  at weak reinforcement using the estimates of [10, 8]. Using the polynomial estimate provided by Merkl and Rolles, [16], we are able to prove recurrence of ERRW on  $\mathbb{Z}^2$  for all initial constant weights, hence giving a full answer to the question of Diaconis.

## 2. STATEMENTS OF THE RESULTS

**2.1. Representation of the VRJP on infinite graphs.** Let  $\mathcal{G} = (V, E)$  be a non oriented, locally finite, connected graph without trivial loop. For  $i, j \in V$ , write  $i \sim j$  if  $i$  is a neighbor of  $j$ . For each edge  $e = \{i, j\} \in E$ , we associate  $W_{i,j} > 0$ , some positive real number as the conductance of  $e$ . We write  $d_{\mathcal{G}}$  for the graph distance in  $\mathcal{G}$ , and for two subsets  $U, U'$  of  $V$ , define  $d_{\mathcal{G}}(U, U') = \inf_{i \in U, j \in U'} d_{\mathcal{G}}(i, j)$ .

Convention : We adopt the notation  $\sum_{i \sim j}$  for the sum on all non-oriented edges  $\{i, j\}$ , counting only once each edge.

**Proposition 1.** *There exists a family of positive random variables  $(\beta_i)_{i \in V}$ , such that for any finite subset  $U \subset V$ , and  $(\lambda_i)_{i \in U} \in \mathbb{R}_+^U$*

$$\mathbb{E} \left( e^{-\sum_{i \in U} \lambda_i \beta_i} \right) = e^{-\sum_{i \sim j, i, j \in U} W_{i,j} (\sqrt{(1+\lambda_i)(1+\lambda_j)} - 1) - \sum_{i \sim j, i \in U, j \notin U} W_{i,j} (\sqrt{1+\lambda_i} - 1)} \frac{1}{\prod_{i \in U} \sqrt{1+\lambda_i}}.$$

*In particular,  $(\beta_i)_{i \in V}$  has the following properties*

- *It is 1-dependent : if  $U, U' \subset V$  are such that  $d_{\mathcal{G}}(U, U') \geq 2$ , then  $(\beta_i)_{i \in U}$  and  $(\beta_j)_{j \in U'}$  are independent.*
- *The marginal  $\beta_i$  is such that  $\frac{1}{2\beta_i}$  is an Inverse Gaussian with parameter  $(\frac{1}{W_i}, 1)$  where  $W_i = \sum_{j \sim i} W_{i,j}$ .*

We denote by  $\nu_V^W(d\beta)$  its distribution.

**Remark 1.** This random field extends to infinite graphs the random field defined in [20]. On finite graphs, its law is explicit, cf [20], Theorem 1, and Theorem C below.

We call *path* in  $\mathcal{G}$  from  $i$  to  $j$  a finite sequence  $\sigma = (\sigma_0, \dots, \sigma_m)$  in  $V$  such that  $\sigma_0 = i$ ,  $\sigma_m = j$  and  $\sigma_k \sim \sigma_{k+1}$ , for  $k = 0, \dots, m-1$ . The length of  $\sigma$  is defined by  $|\sigma| = m$ . For such a path we define

$$(2.1) \quad W_\sigma = \prod_{k=0}^{m-1} W_{\sigma_k, \sigma_{k+1}}, \quad (2\beta)_\sigma = \prod_{k=0}^m (2\beta_{\sigma_k}), \quad (2\beta)_\sigma^- = \prod_{k=0}^{m-1} (2\beta_{\sigma_k}).$$

For the trivial path  $\sigma = (\sigma_0)$ , we define  $W_\sigma = 1$ ,  $(2\beta)_\sigma = 2\beta_{\sigma_0}$ ,  $(2\beta)_\sigma^- = 1$ .

Let  $V_n$  be an increasing sequence of finite connected subsets of  $V$  such that

$$\bigcup_{n=0}^\infty V_n = V.$$

For  $i, j \in V_n$ , we denote by  $\mathcal{P}_{i,j}^{(n)}$  the set of paths  $\sigma$  in  $V_n$  going from  $i$  to  $j$ . Similarly, we denote by  $\bar{\mathcal{P}}_i^{(n)}$ , the set of paths  $\sigma = (\sigma_0, \dots, \sigma_m)$  from  $i \in V_n$  to a point  $j \notin V_n$  and  $\sigma_0, \dots, \sigma_{m-1}$  in  $V_n$ .

**Definition 1.** We define for  $i, j$  in  $V$

$$\hat{G}^{(n)}(i, j) = \begin{cases} \sum_{\sigma \in \mathcal{P}_{i,j}^{(n)}} \frac{W_\sigma}{(2\beta)_\sigma}, & \text{if } i, j \text{ are in } V_n, \\ 0, & \text{otherwise.} \end{cases}$$

Besides, we define for  $i \in V$

$$\psi^{(n)}(i) = \begin{cases} \sum_{\sigma \in \bar{\mathcal{P}}_i^{(n)}} \frac{W_\sigma}{(2\beta)_\sigma}, & \text{if } i \text{ is in } V_n, \\ 1, & \text{otherwise.} \end{cases}$$

Recall the VRJP and its time-changed  $(Z_t)$  defined in the introduction. Our main theorem is the following.

**Theorem 1.** (i) The sequence  $\hat{G}^{(n)}(i, j)$  converges a.s. to a finite random variable

$$\hat{G}(i, j) = \lim_{n \rightarrow \infty} \hat{G}^{(n)}(i, j).$$

(ii) Let  $\mathcal{F}_n$  be the  $\sigma$ -field generated by  $(\beta_i)_{i \in V_n}$ . For all  $i \in V$ ,  $\psi^{(n)}(i)$  is a positive  $\mathcal{F}_n$ -martingale. It converges a.s. to an integrable  $\beta$ -mesurable random variable  $\psi(i)$ . The random field  $(\psi(i))_{i \in V}$  does not depend on the choice of the increasing sequence  $(V_n)$ . Moreover, the quadratic variation of the vectorial martingale  $(\psi^{(n)}(i))_{i \in V}$  is given by

$$\langle \psi(i), \psi(j) \rangle_n = \hat{G}^{(n)}(i, j).$$

In particular,  $\psi^{(n)}(i)$  is bounded in  $L^2$  if and only if  $\mathbb{E}(\hat{G}(i, j)) < \infty$ .

(iii) Let  $\gamma$  be a random variable independent of the field  $(\beta_j)_{j \in V}$  and with law  $\text{Gamma}(\frac{1}{2}, 1)$  (that is, with density  $\mathbf{1}_{\gamma > 0} \frac{1}{\sqrt{\pi\gamma}} e^{-\gamma}$ ). Define

$$G(i, j) = \hat{G}(i, j) + \frac{1}{2} \gamma^{-1} \psi(i) \psi(j),$$

Then the time changed VRJP  $(Z_t)$  on  $V$  with conductances  $(W_{i,j})$  starting from  $i_0$ , is a mixture of Markov Jump processes with jump rates from  $i$  to  $j$

$$(2.2) \quad \frac{1}{2} W_{i,j} \frac{G(i_0, j)}{G(i_0, i)}$$

We denote  $P_x^{\beta, \gamma, i_0}$  the law of Markov jump process which jumps from  $i$  to  $j$  at rate (2.2) starting from  $x$ . Hence, it means that

$$\mathbb{P}_{i_0}^{VRJP}(\cdot) = \int P_{i_0}^{\beta, \gamma, i_0}(\cdot) \nu_V^W(d\beta) \frac{\mathbb{1}_{\gamma > 0}}{\sqrt{\pi\gamma}} e^{-\gamma} d\gamma.$$

(iv) We have a.s.

- The Markov process  $P^{\beta, \gamma, i_0}$  is transient if and only if  $\psi(i) > 0$  for all  $i \in V$ ,
- The Markov process  $P^{\beta, \gamma, i_0}$  is recurrent if and only if  $\psi(i) = 0$  for all  $i \in V$ .

**Notations .** We denote  $\nu_V^W(d\beta, d\gamma) = d\nu_V^W(d\beta) \otimes \frac{\mathbb{1}_{\gamma > 0}}{\sqrt{\pi\gamma}} e^{-\gamma} d\gamma$  the joint law of  $(\beta, \gamma)$ . We also set

$$u(i, j) = \log(G(i, j)) - \log(G(i, i))$$

so that the jumping rates (2.2) can be expressed by

$$\frac{1}{2} W_{i,j} \frac{G(i_0, j)}{G(i_0, i)} = \frac{1}{2} W_{i,j} e^{u(i_0, j) - u(i_0, i)}.$$

**Remark 2.** When the VRJP is recurrent,  $G = \hat{G}$ , and the VRJP can be represented by a  $(\beta_j)_{j \in V}$ -measurable random field. When the VRJP is transient, it is remarkable that the representation involves an extra random variable  $\gamma$ , which is independent of the field  $(\beta_j)$ .

**Remark 3.** The representation (2.2) extends to infinite graphs the representation provided in [20], Theorem 2, for finite graphs. An interesting new feature appears in the transient regime, where the generalized eigenfunction  $\psi$  and the extra  $\gamma$  random variable enters the expression of  $G(i, j)$ . As it appears in the proof, the eigenfunction  $\psi$  can be interpreted as the mixing field of a VRJP starting from infinity.

Let  $\tilde{Z}_n$  be the discrete time process associated with  $(Z_t)$ . Clearly it is a mixture of Markov chains, with conductances

$$W_{i,j} G(i_0, i) G(i_0, j).$$

Let us denote  $\tau_{i_0}^+ = \inf\{n \geq 1, \tilde{Z}_n = i_0\}$ , the first return time to  $i_0$  by  $(\tilde{Z}_n)$ . The point (iv) of the previous theorem is in fact a consequence of the following more precise assertion.

**Proposition 2.** We have,

$$P_i^{\beta, \gamma, i_0}(\tau_{i_0}^+ = \infty) = \begin{cases} \frac{\psi(i_0)^2}{4\gamma \tilde{\beta}_{i_0} \tilde{G}(i_0, i_0) G(i_0, i_0)} & i = i_0 \\ \frac{\psi(i_0)}{2\gamma} \frac{\tilde{G}(i_0, i_0) \psi(i) - \tilde{G}(i_0, i) \psi(i_0)}{\tilde{G}(i_0, i_0) G(i_0, i)} & i \neq i_0 \end{cases}$$

where  $\tilde{\beta}_{i_0} = \sum_{j \sim i_0} \frac{1}{2} W_{i_0, j} \frac{G(i_0, j)}{G(i_0, i_0)}$ . In particular,  $\psi(i_0) = 0$  if and only if  $P_{i_0}^{\beta, \gamma, i_0}(\tau_{i_0}^+ = \infty) = 0$ .

Using Doob's  $h$  transform, the law of the process  $(Z_t)$  conditioned on the event  $\{\tau_0^+ < \infty\}$  or  $\{\tau_0^+ = \infty\}$  can be computed and takes a rather nice form, both in the annealed and quenched cases. We provide these formulae in Section 7.

A natural question that emerges from point (iv) of the theorem is that of a 0-1 law for transience/recurrence. We do not have a general answer but we have an answer in the case of

vertex transitive graphs of conductances. We say that  $(\mathcal{G}, W)$  is vertex transitive if the group of automorphisms of  $\mathcal{G}$  that leaves invariant  $(W_{i,j})$  is transitive on vertices. In particular, it is the case for the cubical graph  $\mathbb{Z}^d$  with constant conductances  $W_{i,j} = W$ . Denote by  $\mathcal{A}$  the group of automorphisms that leave invariant  $W$ .

**Proposition 3.** *If  $(\mathcal{G}, W)$  is vertex transitive and  $\mathcal{G}$  infinite, then under  $\nu_V^W$ ,  $\beta$ ,  $\psi$ ,  $\hat{G}$  are stationary and ergodic for the group of transformations  $\mathcal{A}$ . Moreover, the VRJP is either recurrent or transient, i.e.*

$$\mathbb{P}_{i_0}^{\text{VRJP}}(\text{ every vertex is visited i.o. }) = 1 \text{ or } \mathbb{P}_{i_0}^{\text{VRJP}}(\text{ every vertex is visited f.o. }) = 1.$$

In the first case  $\psi(i) = 0$  for all  $i \in V$ , a.s., in the second case  $\psi(i) > 0$  for all  $i \in V$ , a.s.

N.B : The action of  $\mathcal{A}$  on  $\hat{G}$  is  $(\tau\hat{G})(i, j) = \hat{G}(\tau i, \tau j)$  for  $\tau \in \mathcal{A}$ .

**2.2. Relation with random Schrödinger operators.** Let us now relate Theorem 1 to the properties of the Schrödinger operator associated with the random field  $(\beta_j)$ . Define the operator  $P = (P_{i,j})_{i,j \in V}$  by

$$P_{i,j} = \begin{cases} W_{i,j}, & \text{if } i \sim j, \\ 0, & \text{otherwise.} \end{cases}$$

Then, we consider the Schrödinger operator on  $\mathcal{G}$

$$H_\beta = -P + 2\beta,$$

where  $\beta$  represents the operator of multiplication by the field  $(\beta_j)$ .

**Theorem 2.** (i) *The spectrum of  $H_\beta$  is included in  $[0, \infty)$*

(ii) *The operator  $\hat{G}$  is the inverse of  $H_\beta$  in the following sense : for all  $i, j \in V$ , a.s.*

$$\hat{G}(i, j) = \lim_{\epsilon > 0, \epsilon \rightarrow 0} (H_\beta + \epsilon)^{-1}(i, j).$$

(iii) *We have  $(H_\beta\psi)(i) = 0$  a.s. for all  $i \in V$ .*

(iv) *In the case of the grid  $\mathbb{Z}^d$  and when  $W_{i,j} = W$  is constant,  $\hat{G}$  and  $\psi$  are stationary ergodic for the spacial shift. Moreover, in the transient case,  $\psi$  is a positive generalized eigenfunction with eigenvalue 0 in the sense that  $H_\beta\psi = 0$  and  $\psi$  has at most polynomial growth, i.e. there exists  $C > 0$  and  $p \geq 0$  such that for all  $i \in \mathbb{Z}^d$ , a.s.*

$$|\psi(i)| \leq C\|i\|^p.$$

**2.3. Functional central limit theorem.** Consider the VRJP on  $\mathbb{Z}^d$ ,  $d \geq 3$ , and  $W_{i,j} = W$  for all  $i, j$ . We prove a functional central limit theorem for the discrete time process  $(\tilde{Z}_n)$  at weak reinforcement (i.e. for  $W$  large enough).

**Theorem 3.** *Consider the discrete time VRJP  $(\tilde{Z}_n)_{n \geq 0}$  on  $\mathbb{Z}^d$ ,  $d \geq 3$ , with constant  $W_{i,j} = W$ . Denote*

$$B_t^{(n)} = \frac{\tilde{Z}_{[nt]}}{\sqrt{n}}.$$

*There exists  $\lambda_2 > 0$  such that if  $W > \lambda_2$ , the discrete time VRJP  $(\tilde{Z}_n)$  satisfies a functional central limit theorem, i.e. under  $\mathbb{P}_0^{\text{VRJP}}$ ,  $B_t^{(n)}$  converges in law (for the Skorokhod topology) to a  $d$ -dimensional Brownian motion  $B_t$  with non degenerate isotropic diffusion matrix  $\sigma^2 \text{Id}$ , for some  $0 < \sigma^2 < \infty$ .*



**2.4. Consequences for the Edge Reinforced Random Walk (ERRW).** The Edge Reinforced Random Walk (ERRW) is a famous discrete time process introduced in 1986 by Coppersmith and Diaconis, [4, 11].

Endow the edges of the graph by some positive weights  $(a_e)_{e \in E}$ . Let  $(X_n)_{n \in \mathbb{N}}$  be a random process that takes values in  $V$ , and let  $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$  be the filtration of its past. For any  $e \in E$ ,  $n \in \mathbb{N}$ , let

$$(2.3) \quad N_n(e) = a_e + \sum_{k=1}^n \mathbb{1}_{\{\{X_{k-1}, X_k\} = e\}}$$

be the number of crossings of the (non-directed) edge  $e$  up to time  $n$  plus the initial weight  $a_e$ .

Then  $(X_n)_{n \in \mathbb{N}}$  is called Edge Reinforced Random Walk (ERRW) with starting point  $i_0 \in V$  and weights  $(a_e)_{e \in E}$ , if  $X_0 = i_0$  and, for all  $n \in \mathbb{N}$ ,

$$(2.4) \quad \mathbb{P}(X_{n+1} = j \mid \mathcal{F}_n) = \mathbb{1}_{\{j \sim X_n\}} \frac{N_n(\{X_n, j\})}{\sum_{k \sim X_n} N_n(\{X_n, k\})}.$$

We denote by  $\mathbb{P}_{i_0}^{ERRW}$  the law of the ERRW starting from the initial vertex  $i_0$ .

Important progress have been done in the last ten years in the understanding of this process, cf e.g. [1, 8, 16, 18]. In particular, it was proved in 2012 by Sabot, Tarrès, [18], and Angel, Crawford, Kozma, [1], on any graph with bounded degree at strong reinforcement (i.e. for  $a_e < \tilde{\lambda}_0$  for some fixed  $\tilde{\lambda}_0 > 0$ ) that the ERRW is a mixture of positive recurrent Markov chains. It was proved by Disertori, Sabot, Tarrès [8] that on  $\mathbb{Z}^d$ ,  $d \geq 3$ , the ERRW is transient at weak reinforcement (i.e. for  $a_e > \tilde{\lambda}_1$  for some fixed  $\tilde{\lambda}_1 < \infty$ ).

From Theorem 1 of [18], we know that the ERRW has the law of a VRJP in independent conductances. More precisely, consider  $(W_e)_{e \in E}$  as independent random variables with law  $\text{Gamma}(a_e)$ . Consider the VRJP in conductances  $(W_e)_{e \in E}$  and its underlying discrete time process  $(\tilde{Y}_n)$ . Then the annealed law of  $(\tilde{Y}_n)$  (after expectation with respect to  $W$ ) is that of the ERRW  $(X_n)$  with initial weights  $(a_e)$ . Hence, we can apply Theorem 1 at fixed  $W$  and then integrate on  $W$ . We thus consider the joint law  $\tilde{\nu}_V^a(dW, d\beta, d\gamma)$  of  $W, \beta, \gamma$  defined for any test function  $F$  by

$$\int F(W, \beta, \gamma) \tilde{\nu}_V^a(dW, d\beta, d\gamma) = \mathbb{E} \left( \int F(W, \beta, \gamma) \nu_V^W(d\beta, d\gamma) \right),$$

where the expectation is with respect to the random variables  $(W)$ . We simply denote by  $\tilde{\nu}_V^a(dW, d\beta)$ ,  $\tilde{\nu}_V^a(d\beta)$  the corresponding marginals. From Theorem 1, we see that the ERRW starting from  $i_0$  is a mixture of reversible Markov chain with conductances

$$(2.5) \quad x_{i,j} = W_{i,j} G(i_0, i) G(i_0, j),$$

where  $G$  is defined in Theorem 1, and  $(W, \beta, \gamma)$  are distributed according to  $\tilde{\nu}_V^a(dW, d\beta, d\gamma)$ .

An important point is that we keep the 1-dependence of the field  $\beta$ , after expectation with respect to  $W$ .

**Proposition 4.** *Under  $\tilde{\nu}_V^a(d\beta)$ ,  $(\beta_j)_{j \in V}$  is 1-dependent : if  $U, U' \subset V$  are such that  $d_G(U, U') \geq 2$ , then  $(\beta_i)_{i \in U}$  and  $(\beta_j)_{j \in U'}$  are independent.*

*Proof.* Indeed, from Proposition 1, the Laplace transform of  $(\beta_i)_{i \in U}$  only involves the conductances  $W_{i,j}$  for  $i$  or  $j$  in  $U$ . This implies that the joint Laplace transform of  $(\beta_i)_{i \in U}$  and

$(\beta_i)_{i \in U'}$  is still the product of Laplace transforms even after taking expectation with respect to the random variables  $(W_e)$ .  $\square$

This yields a counterpart of Proposition 3 for the ERRW.

**Proposition 5.** *Assume  $(\mathcal{G}, (a_{i,j}))$  is vertex transitive with automorphism group  $\mathcal{A}$ , and  $\mathcal{G}$  infinite. Then under  $\tilde{\nu}_V^a$ ,  $W$ ,  $\beta$ ,  $\psi$ ,  $\hat{G}$  are stationary and ergodic for the group of transformations  $\mathcal{A}$ . Moreover, the ERRW is either recurrent or transient, i.e.*

$$\mathbb{P}_{i_0}^{ERRW}(\text{every vertex is visited i.o.}) = 1, \text{ or } \mathbb{P}_{i_0}^{ERRW}(\text{every vertex is visited f.o.}) = 1.$$

In the first case  $\psi(i) = 0$  for all  $i \in V$ , a.s., in the second case  $\psi(i) > 0$  for all  $i \in V$ , a.s.

N.B : The action of  $\mathcal{A}$  on  $\hat{G}$  and  $W$  is  $(\tau\hat{G})(i, j) = \hat{G}(\tau i, \tau j)$ ,  $\tau W_{i,j} = W_{\tau i, \tau j}$  for  $\tau \in \mathcal{A}$ .

**Remark 4.** In [15], it was proved on infinite graphs that the ERRW is a mixture of Markov chains, obtained as a weak limit of the mixing measure of the ERRW on finite approximating graphs. The difference in the representation we give in (2.5) is that the random variables  $\psi$ ,  $\hat{G}$  are obtained as almost sure limits and hence are measurable functions of the random variables  $\beta$ . This yields stationarity and ergodicity, which are the key ingredients in the 0-1 law, and in forthcoming Theorems 4 and 5.

**Remark 5.** It seems that this 0-1 law is new, both for the VRJP and the ERRW. In [15], it was proved that if the ERRW comes back with probability 1 to its starting point then it visits infinitely often all points, a.s., which is a weaker result. This was proved using the representation of the ERRW as mixture of Markov chains of [15]. A short proof of that result can also be given, cf [21].

We now give a counterpart of Theorem 3 for the ERRW. It is a consequence of Theorem 1 and of the delocalization result proved by Disertori, Sabot, Tarrès in [8].

**Theorem 4.** *Consider the ERRW  $(X_n)_{n \geq 0}$  on  $\mathbb{Z}^d$ ,  $d \geq 3$ , with constant weights  $a_{i,j} = a$ . Denote*

$$B_t^{(n)} = \frac{X_{[nt]}}{\sqrt{n}}.$$

*There exists  $\tilde{\lambda}_2 > 0$  such that if  $a > \tilde{\lambda}_2$ , the ERRW satisfies a functional central limit theorem, i.e. under  $\mathbb{P}_0^{ERRW}$ ,  $(B_t^{(n)})$  converges in law (for the Skorokhod topology) to a  $d$ -dimensional Brownian motion  $(B_t)$  with non degenerate isotropic diffusion matrix  $\sigma^2 Id$ , for some  $0 < \sigma^2 < \infty$ .*

Finally, we can deduce recurrence of the ERRW in dimension 2 from Theorem 1, Proposition 5 and the estimates obtained by Merkl Rolles in [14, 16]<sup>1</sup>.

**Theorem 5.** *The ERRW  $(X_n)_{n \geq 0}$  on  $\mathbb{Z}^2$  with constant weights  $a_{i,j} = a$  is a.s. recurrent, i.e.*

$$\mathbb{P}_0^{ERRW}(\text{every vertex is visited infinitely often}) = 1.$$

In [14, 16], Merkl and Rolles proved polynomial decrease of the type

$$(2.6) \quad \mathbb{E} \left( \left( \frac{x_\ell}{x_0} \right)^{\frac{1}{4}} \right) \leq c(a) |\ell|^{-\xi(a)},$$

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<sup>1</sup>We are grateful to Franz Merkl and Silke Rolles for a useful discussion on that subject



for some constants  $c(a) > 0$ ,  $\xi(a) > 0$ , depending only on  $a$ , and where  $x_\ell$  is the conductance at the site  $\ell$  for the mixing measure of the ERRW, uniformly for a sequence of finite approximating graphs. When  $0 < \xi < 1$ , it does not give by itself enough information to prove recurrence. It was used in the case of a diluted 2-dimensional graphs to prove positive recurrent at strong reinforcement. The extra information given by the representation (2.5) and the stationarity of  $\psi$ , implies that the polynomial estimate (2.6) is incompatible with  $\psi(i) > 0$  and hence is incompatible with transience. Detailed arguments are provided in Section 8.

**Remark 6.** *We expect similarly that the 2-dimensional VRJP with constant conductances  $W_{i,j} = W > 0$  is recurrent. This would be implied by an estimate of the type (2.6) for the mixing field of the VRJP, which is still not available. More precisely, we can see from the proof of Theorem 5 in Section 8, that recurrence of the 2-dimensional VRJP would be implied by Theorem 1, Proposition 3, and an estimate of the type*

$$\mathbb{E} \left( e^{\eta(u_\ell - u_0)} \right) \leq \epsilon(|\ell|_\infty),$$

for  $\eta > 0$  and  $\epsilon(n)$  a positive function such that  $\lim_{n \rightarrow \infty} \epsilon(n) = 0$ , where  $(u_j)$  is the mixing field of the VRJP starting from 0 (cf Theorem B) on finite boxes with wired boundary condition as in Section 4.2. We learned from G. Kozma and R. Peled that they have a proof of such an estimate.

**2.5. Open questions.** The most important question certainly concerns the relation between the properties of the VRJP and the spectral properties of the random Schrödinger operator  $H_\beta$ . For example on  $\mathbb{Z}^d$  with constant weights  $W_{i,j} = W$ , is recurrence/transience of the VRJP related to the localized/delocalized regimes of  $H_\beta$ ? A more precise question would be : does the transient regime of the VRJP coincide with the existence of extended states at least at the bottom of the spectrum of  $H_\beta$ ? It might at first seem inconsistent to expect extended states at the bottom of the spectrum since the Anderson model with i.i.d. potential is expected to be localized at the edges of the spectrum (a fact which is proved in several cases). But this localization is a consequence of Lifchitz tails, and there are good reasons to expect that Lifchitz tails fail for the potential  $\beta$ , which is not i.i.d. but 1-dependent. Indeed, the bottom of the spectrum of  $H_\beta$  is 0, it does not coincide with the minimum of the support of the distribution of  $2\beta$  translated by the spectrum of  $-P$ , as it is the case for i.i.d. potential. In fact, on a finite set, the minimum of the spectrum is reached on the set  $\det(2\beta - P) = 0$  which is a set of codimension 1, hence it is "big".

Another natural question concerns the uniform integrability of the martingale  $\psi^{(n)}(i)$ . Let us ask a more precise question : is it true (at least for  $\mathbb{Z}^d$  with constant weights) that transience of the VRJP implies that the martingale  $\psi^{(n)}(i)$  is bounded in  $L^2$ ? It is quite natural to expect such a property from relation (5.2) since  $\hat{G}^{(n)}(i, i)$  appears to be the quadratic variation of  $\psi^{(n)}(i)$ . This would have several consequences. Firstly, it would imply that in dimension  $d \geq 3$ , the VRJP satisfies a functional central limit theorem as soon as the VRJP is transient, by the same argument as that of the proof of Theorem 3. It would also imply directly that the VRJP is recurrent as soon as the reversible Markov chain in conductances  $(W_{i,j})$  is recurrent, if the group of automorphisms of  $(\mathcal{G}, W)$  is transitive. Indeed, assume that the property is true and the VRJP is transient. By Theorem 1, the discrete time process  $(\tilde{Z}_n)$  would be represented as a mixture of reversible Markov chains

with conductances  $W_{i,j}G(0,i)G(0,j)$ . It is rather easy (cf Remark 15) to show that

$$\frac{\hat{G}(0,i)}{\hat{G}(0,0)} \leq \frac{\psi(i)}{\psi(0)}.$$

Hence,  $(\tilde{Z}_n)$  is equivalently a mixture of Markov chains with conductances

$$\frac{\psi(0)^2}{G(0,0)^2} W_{i,j} G(0,i) G(0,j) \leq W_{i,j} \psi(i) \psi(j).$$

But  $(\psi(i))$  is stationary ergodic, if  $\psi_0$  is squared integrable, we would have

$$E(W_{i,j} \psi(i) \psi(j)) \leq C W_{i,j}$$

for some constant  $C > 0$ . Usual arguments imply that the Markov chain in conductance  $W_{i,j} \psi(i) \psi(j)$  is recurrent if the Markov chain in conductances  $(W_{i,j})$  is recurrent (cf e.g. Exercice 2.75, [13]). We arrive at a contradiction.

**2.6. Organization of the paper.** In Section 3, we gather results for finite graphs, in particular we recall the main results of [20]. In Section 4, we define the important notion of restriction with wired boundary condition and the compatibility property. Section 5 is the key step in the paper where the martingale property is proved. In Section 6, we prove Theorem 1, Propositions 2 and 3 and Theorem 2. In Section 7, we provide extra computations of  $h$ -transforms of the quenched and annealed VRJP. Section 8, we prove recurrence of ERRW in dimension 2 for all initial weights. In Section 9, we prove functional central limit theorems for the VRJP and the ERRW, Theorems 3 and 4.

### 3. THE RANDOM POTENTIAL $\beta$ ON FINITE GRAPHS

We gather in this section several results for finite graphs.

**3.1. The field  $\beta$  on finite graphs and relation to the VRJP.** In this subsection we consider the case where  $\mathcal{G} = (V, E)$  is a finite graph. Recall that every non oriented edge  $e = \{i, j\}$  is labeled with a positive real number  $W_e = W_{i,j}$ . Firstly, we recall Theorem 1 from [20], which gives the density of  $\beta$  on any finite graph.

**Theorem A** ([20], Theorem 1). *Let  $\mathcal{G} = (V, E)$  be a  $(W_e)$  weighted finite graph as above. The measure below is a probability on  $(\mathbb{R}_+)^V$  :*

$$(3.1) \quad \nu_V^W(d\beta) := \mathbf{1}_{H_\beta > 0} \left( \frac{2}{\pi} \right)^{|V|/2} \exp \left( - \sum_{i \in V} \beta_i + \sum_{e \in E} W_e \right) \frac{d\beta_V}{\sqrt{\det H_\beta}}$$

with  $d\beta_V = \prod_{i \in V} d\beta_i$ , and where  $H_\beta$  is the Schrödinger operator on  $\mathcal{G}$  :  $H_\beta = 2\beta - P$  where  $P$  is the adjacency matrix of the undirected graph  $\mathcal{G}$  with weight  $(W_e)$ , in other words,  $H_\beta$  is the matrix with coefficients

$$H_\beta(i, j) = \begin{cases} 2\beta_i, & i = j, \\ -W_{i,j}, & i \neq j, i \sim j, \\ 0, & \text{otherwise.} \end{cases}$$

If  $(\beta_i, i \in V)$  is  $\nu_V^W$  distributed, then, the Laplace transform of  $(\beta_i)$  is

$$(3.2) \quad \int e^{-\langle \lambda, \beta \rangle} \nu_V^W(d\beta) = \exp \left( - \sum_{i \sim j} W_{i,j} (\sqrt{(\lambda_i + 1)(\lambda_j + 1)} - 1) \right) \prod_{i \in V} \frac{1}{\sqrt{\lambda_i + 1}}.$$

for all  $(\lambda_i) \in \mathbb{R}_+^V$ .

The field  $\beta$  is closely related to the VRJP, as shown in the next two theorems. Consider the VRJP  $(Y_t)$  on  $\mathcal{G}$  with weight  $(W_{i,j})$  and initial local times 1, starting at  $i_0 \in V$ . In [18], it is shown that the time changed process  $Z_t = Y_{D^{-1}}(t)$  (recall from (1.1) that  $D(t) = \sum_{i \in V} (L_i^2(t) - 1)$ ) is a mixture of Markov jump processes, more precisely:

**Theorem B** ([18], Theorem 2). *Assume  $V$  finite. The following measure is a probability distribution on the set  $\{(u_i)_{i \in V} \in \mathbb{R}^V, u_{i_0} = 0\}$ :*

$$(3.3) \quad \mathcal{Q}_{i_0}^W(du) = \frac{1}{\sqrt{2\pi}^{|V|-1}} \exp \left( - \sum_{i \in V} u_i - \sum_{i \sim j} W_{i,j} (\cosh(u_i - u_j) - 1) \right) \sqrt{D(W, u)} du_{V \setminus \{i_0\}}$$

where  $du_{V \setminus \{i_0\}} = \prod_{i \in V \setminus \{i_0\}} du_i$  and

$$D(W, u) = \sum_{T \in \mathcal{T}} \prod_{\{i,j\} \in T} W_{i,j} e^{u_i + u_j}$$

The sum is over  $\mathcal{T}$ , the set of spanning trees of the graph  $\mathcal{G}$ .

The law of the time changed VRJP  $(Z_t)$  starting at  $i_0$  is a mixture of Markov jump processes starting at  $i_0$ , with jump rate  $\frac{1}{2} W_{i,j} e^{u_j - u_i}$  from  $i$  to  $j$ , when  $(u_i)$  is distributed according to  $\mathcal{Q}_{i_0}^W(du)$ .

**Remark 7.** By the matrix-tree theorem,  $D(W, u)$  is any diagonal minor of the  $|V| \times |V|$  matrix  $(m_{i,j})$  with coefficients

$$m_{i,j} = \begin{cases} 0, & \text{if } i \not\sim j, i \neq j \\ -W_{i,j} e^{u_i + u_j}, & \text{if } i \sim j, i \neq j \\ \sum_{k \in V, k \sim i} W_{i,k} e^{u_i + u_k}, & \text{if } i = j \end{cases}$$

**Remark 8.** The probability measure  $\mathcal{Q}_{i_0}^W(du)$  appeared previously to [18] in a rather different context in the work of Disertori, Spencer, Zirnbauer, [10]. In particular, the fact that  $\mathcal{Q}_{i_0}^W(du)$  is a probability measure was proved there as a consequence of a Berezin identity applied to a supersymmetric extension of that measure.

On finite graphs, the random environment  $(u_i)$  of the previous theorem can be represented thanks to the Green function of the random potential  $(\beta_i, i \in V)$ . Let us recall Theorem 3 in [20].

**Theorem C** ([20], Proposition 1 and Theorem 3). *Assume  $V$  finite. Let  $(\beta_j)_{j \in V}$  be  $\nu_V^W$  distributed and let  $G = (H_\beta)^{-1}$  be the green function of the Schrödinger operator  $H_\beta$ . We denote*

$$(3.4) \quad e^{u(i,j)} = \frac{G(i,j)}{G(i,i)}.$$

For all  $i_0 \in V$ , we have the following properties

- (i) the random field  $(u(i_0, j))_{j \in V}$  has the distribution  $\mathcal{Q}_{i_0}^W$  of Theorem B,
- (ii)  $(u(i_0, j))_{j \in V}$  is  $(\beta_j)_{j \in V \setminus \{i_0\}}$ -measurable.
- (iii)  $G(i_0, i_0)$  is equal in law to  $\frac{1}{2\gamma}$ , where  $\gamma$  is a gamma random variable with parameter  $(1/2, 1)$ ,
- (iv)  $G(i_0, i_0)$  is independent of  $(\beta_j)_{j \neq i_0}$ , hence independent of the field  $(u(i_0, j))_{j \in V}$ ,
- (v) for all  $i_0 \in V$ ,  $i \in V$

$$(3.5) \quad \beta_i = \frac{1}{2} \sum_{j \sim i} W_{i,j} e^{u(i_0, j) - u(i_0, i)} + \frac{\mathbb{1}_{i=i_0}}{2G(i_0, i_0)}.$$

**Remark 9.** Here we only consider the VRJP with initial local time 1, in fact, the above correspondence between  $\beta$  and VRJP still holds for the process starting with any positive local times  $(\phi_i, i \in V)$ , in such case, there is a corresponding density  $\nu_V^{W, \phi^2}$ , which is defined in [20]. We choose here to normalize the initial local time to 1 since it is equivalent to the general case by a change of time and  $W$ , see [20].

The green function  $G(\cdot, \cdot)$  has a representation as a path sum.

**Proposition 6.** Assume that  $V$  is finite. Let  $\mathcal{P}_{i,j}^V$  be the collection of path in  $V$  from  $i$  to  $j$ , and  $\bar{\mathcal{P}}_{i,j}^V$  be the collection of paths  $\sigma = (\sigma_0 = i, \dots, \sigma_m = j)$  in  $V$  from  $i$  to  $j$  such that  $\sigma_k \neq j, k = 0, \dots, m-1$ . For all  $(\beta_j)_{j \in V} \in \mathbb{R}^V$  such that  $2\beta - P > 0$ , we have, with the notations of Theorem C,

$$(3.6) \quad G(i, j) = \sum_{\sigma \in \mathcal{P}_{i,j}^V} \frac{W_\sigma}{(2\beta)_\sigma}$$

and,

$$(3.7) \quad \exp(u(i, j)) = \sum_{\sigma \in \bar{\mathcal{P}}_{j,i}^V} \frac{W_\sigma}{(2\beta)_\sigma}.$$

*Proof.* Firstly we show that  $\sum_{\sigma \in \mathcal{P}_{i,j}^V} \frac{W_\sigma}{(2\beta)_\sigma}$  converges. Note that  $(2\beta - P) > 0$  is an M-matrix,  $G = (2\beta - P)^{-1}$  is well defined and  $G(i, j) > 0$  for all  $i, j \in V$ . Consider, for  $K \geq 0$

$$G^K(i, j) = \sum_{\sigma \in \mathcal{P}_{i,j}^V, |\sigma| \leq K} \frac{W_\sigma}{(2\beta)_\sigma},$$

It can be shown by recurrence that for any  $K \geq 0$ ,  $G^K(i, j) \leq G(i, j)$ .

- $K = 0$ , as  $\beta_i$  are a.s. strictly positive, for  $i = j$  we have

$$G^0(i, i) = \frac{1}{2\beta_i} \leq G(i, i).$$

(Indeed,  $H_\beta G = \text{Id}$ , hence  $2\beta_i G(i, i) - (PG)(i, i) = 1$  which implies  $2\beta_i G(i, i) \geq 1$ .)

If  $i \neq j$ , then clearly  $G^0(i, j) = 0 \leq G(i, j)$ .

- For the inductive step, note that  $GH_\beta = \text{Id}$  gives for all  $i, j$

$$(3.8) \quad 2\beta_j G(i, j) - \sum_{l \sim j} W_{l,j} G(i, l) = \mathbb{1}_{i=j}.$$

If  $G^K(i, j) \leq G(i, j)$ , then using the previous identity

$$\begin{aligned}
 G^{K+1}(i, j) &= \sum_{\sigma \in \mathcal{P}_{i,j}^V, |\sigma| \leq K+1} \frac{W_\sigma}{(2\beta)_\sigma} \\
 (3.9) \quad &= \frac{\mathbb{1}_{i=j}}{2\beta_j} + \sum_{l \sim j} G^K(i, l) \frac{W_{l,j}}{2\beta_j} \\
 &\leq \frac{\mathbb{1}_{i=j}}{2\beta_j} + \sum_{l \sim j} \frac{W_{l,j}}{2\beta_j} G(i, l) \\
 &= G(i, j).
 \end{aligned}$$

Let us define  $G'(i, j) = \lim_{K \rightarrow \infty} G^K(i, j) = \sum_{\sigma \in \mathcal{P}_{i,j}^V} \frac{W_\sigma}{(2\beta)_\sigma} < \infty$ . Note that  $H_\beta$  is a.s. positive definite, its inverse is uniquely determined, hence it is enough to check the equation  $G'H_\beta = \text{Id}$ . Passing to the limit in the second equality of equation (3.9), gives

$$G'(i, j) = \frac{\mathbb{1}_{i=j}}{2\beta_j} + \sum_{l \sim j} G'(i, l) \frac{W_{l,j}}{2\beta_j}$$

which is equivalent to  $G'H_\beta = 1$ .

For (3.7), note first that  $\sum_{\sigma \in \bar{\mathcal{P}}_{j,i}^V} \frac{W_\sigma}{\beta_\sigma} \leq \beta_i G(j, i) < \infty$  a.s.. A path in  $\mathcal{P}_{j,i}^V$  can be cut at its first visit to  $i$ , turning it into the concatenation of a path in  $\bar{\mathcal{P}}_{j,i}^V$  and a path in  $\mathcal{P}_{i,i}^V$ , and this operation is bijective. It implies that

$$(3.10) \quad \left( \sum_{\sigma \in \bar{\mathcal{P}}_{j,i}^V} \frac{W_\sigma}{(2\beta)_\sigma} \right) G(i, i) = \left( \sum_{\sigma \in \bar{\mathcal{P}}_{j,i}^V} \frac{W_\sigma}{(2\beta)_\sigma} \right) \left( \sum_{\sigma \in \mathcal{P}_{i,i}^V} \frac{W_\sigma}{(2\beta)_\sigma} \right) = \sum_{\sigma \in \mathcal{P}_{j,i}^V} \frac{W_\sigma}{(2\beta)_\sigma} = G(j, i) = G(i, j),$$

hence equation (3.7).  $\square$

**3.2. A priori estimates on  $e^{u(i,j)}$ .** The following proposition is borrowed from [10], Lemma 3. For convenience we give a shorter proof of that estimate based on spanning trees instead of fermionic variables, following the proof of the corresponding result for the ERRW, c.f. [8], Lemma 7.

**Proposition 7.** *Let  $\mathcal{G} = (V, E)$  be a finite graph with edge weights  $(W_{i,j})$ . Fix a vertex  $i_0$ . Let  $\eta > 0$ . If there exists a path  $\sigma = (\sigma_0, \dots, \sigma_K)$  from  $i \in V$  to  $j \in V$  of length  $K$  such that  $W_{\sigma_k, \sigma_{k+1}} \geq 2\eta$  for all  $k = 0, \dots, K-1$ , then*

$$\mathbb{E}(e^{\eta \cosh(u(i_0, j) - u(i_0, i))}) \leq 2^{K/2}$$

where  $u(i_0, j)$  is the mixing field of the VRJP starting at  $i_0$  defined in Theorem C.

*Proof.* We simply write  $u(j)$  for  $u(i_0, j)$  in this proof. By Theorem C, the density of  $(u(i))$  on  $\{(u(i))_{i \in V} \in \mathbb{R}^V, u(i_0) = 0\}$  is

$$\mathcal{Q}_{i_0}^W(du) = \frac{1}{\sqrt{2\pi}^{|V|-1}} \exp\left(-\sum_i u(i) - \sum_{i \sim j} W_{i,j}(\cosh(u(i) - u(j)) - 1)\right) \sqrt{D(W, u)} du_{V \setminus \{i_0\}},$$

with  $du_{V \setminus \{i_0\}} = \prod_{i \neq i_0} du_i$ .

Consider a path  $\sigma_0 = i, \sigma_1, \dots, \sigma_K = j$  as in the statement of the proposition and assume that it is simple. We have

$$(3.11) \quad \cosh(u(i) - u(j)) \leq \sum_{k=1}^K \cosh(u(\sigma_{k-1}) - u(\sigma_k)).$$

Let  $\tilde{W} = W - \eta \sum_{k=1}^K \mathbb{1}_{\{\sigma_{k-1}, \sigma_k\}}$ , (i.e.  $\tilde{W}$  is equal to  $W - \eta$  on the path and unchanged on the complement of the path). By assumption, we have  $\tilde{W}_{i,j} > 0$  on the edges, and for all spanning tree  $T$

$$\begin{aligned} \prod_{\{i,j\} \in T} W_{i,j} e^{u(i)+u(j)} &\leq \left( \prod_{k=1}^K \frac{W_{\sigma_{k-1}, \sigma_k}}{W_{\sigma_{k-1}, \sigma_k} - \eta} \right) \prod_{\{i,j\} \in T} \tilde{W}_{i,j} e^{u(i)+u(j)} \\ &\leq 2^K \prod_{\{i,j\} \in T} \tilde{W}_{i,j} e^{u(i)+u(j)}, \end{aligned}$$

which implies

$$D(W, u) \leq 2^K D(\tilde{W}, u).$$

From (3.11) and the expression of  $\mathcal{Q}_{i_0}^W(du)$ , we deduce that

$$\exp(\eta \cosh(u(i) - u(j))) \mathcal{Q}_{i_0}^W(du) \leq 2^{K/2} \mathcal{Q}_{i_0}^{\tilde{W}}(du).$$

It implies that

$$\mathbb{E}(e^{\eta \cosh(u(i) - u(j))}) = \int e^{\eta \cosh(u(i) - u(j))} \mathcal{Q}_{i_0}^W(du) \leq 2^{K/2} \int \mathcal{Q}_{i_0}^{\tilde{W}}(du) = 2^{K/2}.$$

□

#### 4. THE WIRED BOUNDARY CONDITION AND KOLMOGOROV EXTENSION TO INFINITE GRAPHS

**4.1. Restriction with wired boundary condition.** Our objective is to extend the relations between the VRJP and the  $\beta$  field to the case of infinite graphs. To this end, we need appropriate boundary condition, which turns out to be the wired boundary condition.

**Definition 2.** Let  $\mathcal{G} = (V, E)$  be a connected graph with finite degree at each site, and  $V_1$  a strict finite subset of  $V$ . We define the restriction of  $\mathcal{G}$  to  $V_1$  with wired boundary condition as the graph  $\mathcal{G}_1 = (\tilde{V}_1 = V_1 \cup \{\delta\}, E_1)$  where  $\delta$  is an extra point and

$$E_1 = \{\{i, j\} \in E, \text{ s.t. } i \in V_1, j \in V_1, i \sim j\} \cup \{\{i, \delta\}, i \in V_1 \text{ s.t. } \exists j \notin V_1, i \sim j\}.$$

If  $(W_{i,j})_{\{i,j\} \in E}$  is a set of positive conductances, we define  $(W_{i,j}^{(1)})_{\{i,j\} \in E_1}$  as the set of restricted conductances by

$$\begin{cases} W_{i,j}^{(1)} = W_{i,j}, & \text{if } i, j \in V_1, \{i, j\} \in E_1, \\ W_{i,\delta}^{(1)} = \sum_{j \notin V_1, j \sim i} W_{i,j}, & \text{if } \{i, \delta\} \in E_1, \\ 0, & \text{otherwise.} \end{cases}$$

**Remark 10.** Intuitively, this restriction corresponds to identify all points in  $V \setminus V_1$  to a single point  $\delta$  and to delete the edges connecting points of  $V \setminus V_1$ . The new weights are obtained by summing the weights of the edges identified by this procedure.



The following lemma is fundamental and is the justification for the choice of this notion of restriction.

**Lemma 1.** *Let  $\mathcal{G} = (V, E)$  be a finite graph with conductances  $(W_{i,j})$ . Let  $V_1$  be a strict subset of  $V$  and  $\mathcal{G}_1$  be its restriction with wired boundary condition. Let  $(\beta_j)_{j \in V}$  be distributed according to  $\nu_V^W$  (c.f. Proposition 1). Let  $\beta^{(1)}$  be distributed according to  $\nu_{V_1}^{W^{(1)}}$ . Then*

$$\beta|_{V_1} \stackrel{\text{law}}{=} \beta_{|V_1}^{(1)}.$$

**Remark 11.** *Note that there is no such compatibility relation with the more usual notion of restriction of graph. The wired boundary condition is fundamental and in fact will be responsible for the extra gamma random variable that appears in the representation of the VRJP on the infinite graph.*

*Proof.* Taking  $\lambda_{|V \setminus V_1} = 0$  in Theorem A, the Laplace transform of  $(\beta_i, i \in V_1)$  is

$$(4.1) \quad \mathbb{E} \left( e^{-\sum_{i \in V_1} \lambda_i \beta_i} \right) = \exp \left( - \sum_{i \sim j, i, j \in V_1} W_{i,j} (\sqrt{(1 + \lambda_i)(1 + \lambda_j)} - 1) - \sum_{i \sim j, i \in V_1, j \notin V_1} (W_{i,j} (\sqrt{1 + \lambda_i} - 1)) \right) \prod_{i \in V_1} \frac{1}{\sqrt{1 + \lambda_i}}.$$

Applying Theorem A to the graph  $\mathcal{G}_1$  with  $\lambda_\delta = 0$ , we get

$$(4.2) \quad \mathbb{E} \left( e^{-\sum_{i \in V_1} \lambda_i \beta_i^{(1)}} \right) = \exp \left( - \sum_{i \sim j, i, j \in V_1} W_{i,j}^{(1)} (\sqrt{(1 + \lambda_i)(1 + \lambda_j)} - 1) - \sum_{i \in V_1, i \sim \delta} (W_{i,\delta}^{(1)} (\sqrt{1 + \lambda_i} - 1)) \right) \prod_{i \in V_1} \frac{1}{\sqrt{1 + \lambda_i}}.$$

By definition of  $W_{i,j}^{(1)}$ , these Laplace transforms are equal.  $\square$

**4.2. Kolmogorov extension : proof of Proposition 1.** Let  $\mathcal{G} = (V, E)$  be a connected infinite graph with finite degree at each site with conductances  $(W_{i,j})$ . Let  $(V_n)_{n \geq 1}$  be an increasing sequence of finite strict subsets of  $V$  that exhausts  $V$

$$\cup_n V_n = V.$$

Let  $\mathcal{G}_n = (\tilde{V}_n = V_n \cup \{\delta_n\}, E_n)$  be the restriction of  $\mathcal{G}$  to  $V_n$  with wired boundary condition, and  $(W^{(n)})$  the restricted conductances. By construction, if  $n < m$ , then  $(\mathcal{G}_n, W^{(n)})$  is the restriction with wired boundary condition of  $(\mathcal{G}_m, W^{(m)})$ . Let  $\beta^{(n)}$  be the random field with distribution  $\nu_{\tilde{V}_n}^{W^{(n)}}$ . By Lemma 1, we know that  $(\beta_{|V_n}^{(n)})$  is a compatible sequence of random variables. By Kolmogorov extension theorem, there exists a random field  $(\beta_j)_{j \in V}$ , such that  $\beta_{|V_n} \stackrel{\text{law}}{=} \beta_{|V_n}^{(n)}$ . This immediately implies that  $(\beta)$  has the Laplace transform given in Proposition 1. We denote by  $\nu_V^W$  its law.

Moreover, we can couple the sequence of random variables  $(\beta^{(n)})$  on  $V_n \cup \{\delta_n\}$ , with distribution  $\nu_{\tilde{V}_n}^{W^{(n)}}$ , with  $\beta \sim \nu_V^W$  and an extra independent gamma random variable. Indeed,

let  $\gamma$  be a random variable with distribution  $\text{Gamma}(\frac{1}{2}, 1)$ , independent of  $(\beta) \sim \nu_V^W$ . Define  $\beta^{(n)}$  by

$$(4.3) \quad \beta_{|V_n}^{(n)} = \beta_{|V_n}, \quad \beta_{\delta_n}^{(n)} = \sum_{j \in V_n, j \sim \delta_n} \frac{1}{2} W_{j, \delta_n} e^{u^{(n)}(\delta_n, j)} + \gamma,$$

where  $u^{(n)}$  is the field defined in Theorem C (Recall that  $u^{(n)}(\delta_n, \cdot)$  only depends on  $(\beta^{(n)})_{|V_n}$  and not on  $\beta_{\delta_n}^{(n)}$ ). From Theorem C, it is clear that  $(\beta_j^{(n)})_{j \in \tilde{V}_n}$  follows the law  $\nu_{\tilde{V}_n}^{W^{(n)}}$ . We always consider  $(\beta^{(n)})$  and  $(\beta)$  coupled in such way in the sequel. We denote, as in Theorem 1 iii), by  $\nu_V^W(d\beta, d\gamma)$  the joint law of  $\beta$  and  $\gamma$ .

**4.3. Definition of  $G^{(n)}$  and the relation between  $G^{(n)}$ ,  $\hat{G}^{(n)}$ ,  $\psi^{(n)}$  and  $\gamma$ .** Recall the definition of  $\mathcal{P}_{i,j}^{(n)}$  given in Section 2. It is clear from the definition given in the previous section that  $\mathcal{P}_{i,j}^{(n)}$  coincide with  $\mathcal{P}_{i,j}^{V_n}$  defined in Proposition 6. With the previous definition it implies from the same proposition that on the set  $V_n$  we have

$$(4.4) \quad (\hat{G}^{(n)})_{|V_n \times V_n} = ((H_\beta)_{|V_n \times V_n})^{-1}.$$

Similarly, we clearly have that  $\bar{\mathcal{P}}_i^{(n)}$  defined in Section 2 coincides with the set  $\bar{\mathcal{P}}_{i, \delta_n}^{\tilde{V}_n}$ . This implies that

$$(4.5) \quad \psi^{(n)}(i) = e^{u^{(n)}(\delta_n, i)}$$

when  $i \in V_n$ , where  $u^{(n)}$  corresponds to the field defined in Theorem C from the potential  $\beta^{(n)}$ . (Note that  $u^{(n)}(\delta_n, i)$  only depends on  $\beta_{|V_n}^{(n)} = \beta_{|V_n}$  and not on the value of the potential on  $\delta_n$ ).

Finally, we introduce the matrix  $(G^{(n)}(i, j))_{i, j \in \tilde{V}_n}$  by

$$G^{(n)} = (H_\beta^{(n)})^{-1}.$$

where as usual  $H_\beta^{(n)} = 2\beta^{(n)} - P$  is the  $\tilde{V}_n \times \tilde{V}_n$  Schrödinger operator relative to the potential  $\beta^{(n)}$  on the graph  $\mathcal{G}_n$ , as in Theorem A. From (3.6)

$$(4.6) \quad G^{(n)}(i, j) = \sum_{\sigma \in \mathcal{P}_{i,j}^{\tilde{V}_n}} \frac{W_\sigma}{(2\beta)_\sigma}.$$

It is hence immediate that for  $i$  and  $j$  in  $V_n$ ,

$$(4.7) \quad \hat{G}^{(n)}(i, j) \leq G^{(n)}(i, j),$$

since  $\mathcal{P}_{i,j}^{(n)} = \mathcal{P}_{i,j}^{V_n} \subset \mathcal{P}_{i,j}^{\tilde{V}_n}$  and  $\beta$  are a.s. positive.

**Proposition 8.** *With the previous notations and with the coupling of section 4.2*

$$(4.8) \quad G^{(n)}(\delta_n, \delta_n) = \frac{1}{2\gamma}.$$

Moreover,

$$G^{(n)}(i, j) = \hat{G}^{(n)}(i, j) + \psi^{(n)}(i)\psi^{(n)}(j)G^{(n)}(\delta_n, \delta_n).$$

*Proof.* The first equality is a direct consequence of the special choice for the coupling (4.3) and of the identity (3.5) in Theorem C.

By Proposition 6, we find that

$$\begin{cases} G^{(n)}(i, j) = \sum_{\sigma \in \mathcal{P}_{i,j}^{\tilde{V}_n}} \frac{W_\sigma}{(2\beta)_\sigma} \\ \psi^{(n)}(i) = \frac{G^{(n)}(\delta_n, i)}{G^{(n)}(\delta_n, \delta_n)} \end{cases}$$

Therefore, if we denote  $\mathcal{P}_{i,\delta_n,j}^{\tilde{V}_n}$  the collection of paths on  $\tilde{V}_n$  starting from  $i$ , visiting  $\delta_n$  at least once, and ending at  $j$ , that is,

$$\mathcal{P}_{i,\delta_n,j}^{\tilde{V}_n} = \{\sigma = (\sigma_0, \dots, \sigma_m) \in \mathcal{P}_{i,j}^{\tilde{V}_n}, \text{ such that } \exists 0 \leq k \leq m, \sigma_k = \delta_n\}$$

then

$$\begin{aligned} G^{(n)}(i, j) - \hat{G}^{(n)}(i, j) &= \sum_{\sigma \in \mathcal{P}_{i,\delta_n,j}^{\tilde{V}_n}} \frac{W_\sigma}{(2\beta^{(n)})_\sigma} \\ &= \left( \sum_{\sigma \in \mathcal{P}_{i,\delta_n}^{\tilde{V}_n}} \frac{W_\sigma}{(2\beta^{(n)})_\sigma^-} \right) \cdot \left( \sum_{\sigma \in \mathcal{P}_{\delta_n,j}^{\tilde{V}_n}} \frac{W_\sigma}{(2\beta^{(n)})_\sigma} \right) \\ &= \psi^{(n)}(i) G^{(n)}(\delta_n, j) = \psi^{(n)}(i) \psi^{(n)}(j) G^{(n)}(\delta_n, \delta_n). \end{aligned}$$

□

## 5. THE MARTINGALE PROPERTY

We denote by  $\mathcal{F}_n = \sigma(\beta_i, i \in V_n)$ , the sigma field generated by  $\{\beta_i, i \in V_n\}$ . The following proposition is the key property for the main theorem.

**Proposition 9.** *For all  $n$ ,  $\psi^{(n)}$  has finite moments. Moreover, we have*

$$(5.1) \quad \mathbb{E}(\psi^{(n+1)}(i) | \mathcal{F}_n) = \psi^{(n)}(i), \quad \forall i \in V,$$

and for all  $i, j \in V$ ,

$$(5.2) \quad \mathbb{E}(\psi^{(n+1)}(i) \psi^{(n+1)}(j) - \psi^{(n)}(i) \psi^{(n)}(j) | \mathcal{F}_n) = \mathbb{E}(\hat{G}^{(n+1)}(i, j) - \hat{G}^{(n)}(i, j) | \mathcal{F}_n).$$

**Remark 12.** *In Theorem B, by the substitution  $\tilde{u}(\cdot) = u(\cdot) - \frac{\sum_{i \in V} u(i)}{|V|}$ , where the new variables  $\tilde{u}_{V \setminus \{i_0\}}$  are in the space  $\{\sum_{i \in V} \tilde{u}(i) = 0\}$ , the density becomes*

$$\tilde{\mathcal{Q}}_{i_0}^W(d\tilde{u}) = \frac{1}{\sqrt{2}^{|V|-1}} e^{\tilde{u}(i_0)} e^{-\sum_{i \sim j} W_{i,j} (\cosh(\tilde{u}(i) - \tilde{u}(j)) - 1)} \sqrt{D(W, \tilde{u})} d\tilde{u}_{V \setminus \{i_0\}}.$$

We see from this expression that  $e^{\tilde{u}(i) - \tilde{u}(i_0)} \cdot \tilde{\mathcal{Q}}_{i_0}^W = \tilde{\mathcal{Q}}_i^W$ , hence that  $\int e^{\tilde{u}(i) - \tilde{u}(i_0)} \tilde{\mathcal{Q}}_{i_0}^W(d\tilde{u}) = 1$ . Applied to  $V = \tilde{V}_n$ ,  $i_0 = \delta_n$ , we get  $\mathbb{E}(\psi^{(n)}(i)) = 1$  which is a particular case of (5.1).

The original proof of that property was rather technical (see the second arXiv version of the paper). Some time after the first version of this paper was posted on arXiv, a simpler proof of the martingale property (5.1) was given in [7]. Moreover, using some supersymmetric arguments, the following more general property was proved.

**Lemma 2** ([7]). *Let  $\lambda \in (\mathbb{R}_+)^V$  be a non-negative function on  $V$  with bounded support, then*

$$\mathbb{E} \left( e^{-\langle \lambda, \psi^{(n+1)} \rangle - \frac{1}{2} \langle \lambda, \hat{G}^{(n+1)} \lambda \rangle} \middle| \mathcal{F}_n \right) = e^{-\langle \lambda, \psi^{(n)} \rangle - \frac{1}{2} \langle \lambda, \hat{G}^{(n)} \lambda \rangle},$$

We provide here a different proof of this assertion based on elementary computations on the measures  $\nu_V^W$  on finite sets. It also provides a simpler proof of the original assertion Proposition 9 by differentiating in  $\lambda$ .

**5.1. Marginal and conditional laws of  $\nu_V^W$ .** In this subsection, we suppose that  $\mathcal{G} = (V, E)$  is finite. We state some identities on marginal and conditional laws of the distribution  $\nu_V^W$ , which will be instrumental in the proof of the martingale property in the next subsection.

Let us first remark that the law  $\nu_V^W$  defined in Theorem A can be extended to the case where  $P = (W_{i,j})_{i,j \in V}$  has non zero, diagonal coefficients. Indeed, if some diagonal coefficients of  $P$  are positive, then changing from variables  $(\beta_i)$  to variables  $(\beta_i - \frac{1}{2}W_{i,i})$ , we get the law  $\nu_V^{\tilde{W}}$  where  $(\tilde{W}_{i,j})$  is obtained from  $(W_{i,j})$  by replacing all diagonal entries by 0. While it is not very natural from the point of view of the VRJP to allow non zero diagonal coefficients, it is convenient in this section to allow this possibility since it simplifies the statements about conditional law.

To simplify notations, in the sequel, for any function  $\zeta : V \mapsto \mathbb{R}$  and any subset  $U \subset V$ , we write  $\zeta_U$  for the restriction of  $\zeta$  to the subset  $U$ . We write  $d\beta_U = \prod_{i \in U} d\beta_i$  to denote integration on variables  $\beta_U$ . Similarly, if  $A$  is a  $V \times V$  matrix and  $U \subset V$ ,  $U' \subset V$ , we write  $A_{U,U'}$  for its restriction to the block  $U \times U'$ .

We start by an extension of the family  $\nu_V^W$  due to Letac, [12] (unpublished). We give a proof of this lemma using Theorem A and forthcoming Lemma 4.

**Lemma 3** (Letac, [12]). *Let  $V$  be finite and  $P = (W_{i,j})_{i,j \in V}$  be a symmetric matrix with non-negative coefficients. Let  $(\eta_i)_{i \in V} \in \mathbb{R}_+^V$  be a vector with non-negative coefficients. Then the following distribution on  $\mathbb{R}^V$*

$$(5.3) \quad \begin{aligned} \nu_V^{W,\eta}(d\beta) &:= e^{-\frac{1}{2} \langle \eta, (H_\beta)^{-1} \eta \rangle} e^{\langle \eta, 1 \rangle} \nu_V^W(d\beta) \\ &= \mathbb{1}_{H_\beta > 0} \left( \frac{2}{\pi} \right)^{|V|/2} e^{-\frac{1}{2} \langle 1, H_\beta 1 \rangle - \frac{1}{2} \langle \eta, (H_\beta)^{-1} \eta \rangle} \frac{1}{\sqrt{\det H_\beta}} e^{\langle \eta, 1 \rangle} d\beta \end{aligned}$$

is a probability distribution, where 1 in the scalar products  $\langle 1, H_\beta 1 \rangle$  and  $\langle 1, \eta \rangle$  are to be

understood as the vector  $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ . Its Laplace transform is, for any  $\lambda \in \mathbb{R}_+^V$

$$(5.4) \quad \int e^{-\langle \lambda, \beta \rangle} \nu_V^{W,\eta}(d\beta) = e^{\sum_{i \sim j} W_{i,j} (\sqrt{(1+\lambda_i)(1+\lambda_j)} - 1) - \langle \eta, \sqrt{\lambda+1} - 1 \rangle} \prod_{i \in V} \frac{1}{\sqrt{1+\lambda_i}}$$

where  $\sqrt{\lambda+1} - 1$  should be considered as the vector  $(\sqrt{\lambda_i+1} - 1)_{i \in V}$ .

It appears in the following lemma that this extension is a marginal law of  $\nu_V^W$ , and that marginal and conditional distributions of  $\nu_V^{W,\eta}$  belong to the same family.

**Lemma 4.** *Assume that  $\beta$  is distributed according to  $\nu_V^{W,\eta}$ . Let  $U \subset V$ .*

(i) *Then,  $\beta_U$  is distributed according to  $\nu_U^{W_U, \hat{\eta}}$ , where*

$$(5.5) \quad \hat{\eta} = \eta_U + P_{U,U^c}(1_{U^c}).$$

(ii) Let  $\check{P} = (\check{W}_{i,j})_{i,j \in U}$  and  $\check{\eta} \in (\mathbb{R}_+)^{U^c}$  be the matrix and vector defined by

$$\check{P} = P_{U^c, U^c} + P_{U^c, U} ((H_\beta)_{U, U})^{-1} P_{U, U^c}, \quad \check{\eta} = \eta_{U^c} + P_{U^c, U} ((H_\beta)_{U, U})^{-1} (\eta_U).$$

Then, the law of  $\beta_{U^c}$ , conditionally on  $\beta_U$ , is  $\nu_{U^c}^{\check{W}, \check{\eta}}$ .

**Remark 13.** Note that  $\check{P}$  have non zero diagonal coefficients.

N.B. As we can observe, all the quantities with  $\check{\cdot}$  are relative to vectors or matrices on  $U^c$ , while the quantities with  $\hat{\cdot}$  are relative to vectors or matrices on  $U$ .

**Lemma 5.** Let  $\mathcal{G} = (V, E)$  be a finite connected graph endowed with conductances  $P = (W_{i,j})_{i,j \in V}$ . Let  $(\eta_i)_{i \in V} \in \mathbb{R}_+^V$  be a vector with non-negative coefficients. Let  $U \subset V$ . Let  $(\beta_i)_{i \in V}$  be  $\nu_V^{W, \eta}$ -distributed, define  $\psi = G_\beta \eta$  where  $G_\beta = H_\beta^{-1}$ ; define  $\hat{\eta} = \eta_U + P_{U, U^c} 1_{U^c}$ ,  $\hat{G}^U = ((H_\beta)_{U, U})^{-1}$  and  $\hat{\psi} = \hat{G}^U(\hat{\eta})$ , for any  $\lambda \in \mathbb{R}_+^V$ , we have

$$(5.6) \quad \mathbb{E}_{\nu_V^{W, \eta}}(e^{-\langle \lambda, \psi \rangle - \frac{1}{2} \langle \lambda, G_\beta \lambda \rangle} | \mathcal{F}_U) = e^{-\langle \lambda_U, \hat{\psi} \rangle - \langle \lambda_{U^c}, 1_{U^c} \rangle - \frac{1}{2} \langle \lambda_U, \hat{G}^U \lambda_U \rangle}$$

where  $\mathcal{F}_U = \sigma(\beta_i, i \in U)$ .

*Proof of Lemma 3.* Lemma 3 is implied by (i) of Lemma 4 in the case where  $\eta = 0$  and by [20], Theorem 1 (or Theorem A). Indeed, when  $\eta = 0$ , Lemma 4 (i) implies that if  $\beta \sim \nu_V^W$ , then  $\beta_U$  has distribution  $\nu_U^{W_U, \hat{\eta}}$  with  $\hat{\eta} = P_{U, U^c}(1_{U^c})$ . In particular, it implies that  $\nu_U^{W_U, \hat{\eta}}$  is a probability (one can check that, in the case  $\eta = 0$ , Lemma 3 is not necessary in the proof of Lemma 4, see Remark 14 below). Any  $\hat{\eta} \in (\mathbb{R}_+)^{U^c}$  can be obtained by this procedure by a good choice of  $P_{U, U^c}$ .  $\square$

*Proof of Lemma 4.* The assertions (i) and (ii) are consequences of the same decomposition of the measure  $\nu_V^{W, \eta}$ . It is partially inspired by computations in [12]. We write  $H_\beta$  as block matrix

$$H_\beta = \begin{pmatrix} H_{U, U} & -P_{U, U^c} \\ -P_{U^c, U} & H_{U^c, U^c} \end{pmatrix} \text{ and define } \hat{G}^U = (H_{U, U})^{-1},$$

Now, define the Schur's complement

$$(5.7) \quad \check{H}^{U^c} = H_{U^c, U^c} - P_{U^c, U} \hat{G}^U P_{U, U^c},$$

and

$$\check{G}^{U^c} = (\check{H}^{U^c})^{-1}$$

We have

$$(5.8) \quad H_\beta = \begin{pmatrix} I_U & 0 \\ -P_{U^c, U} \hat{G}^U & I_{U^c} \end{pmatrix} \begin{pmatrix} H_{U, U} & 0 \\ 0 & \check{H}^{U^c} \end{pmatrix} \begin{pmatrix} I_U & -\hat{G}^U P_{U, U^c} \\ 0 & I_{U^c} \end{pmatrix}.$$

Remark that with notations of (ii) we have

$$\check{H}^{U^c} = 2\beta_{U^c} - \check{P},$$

By (5.8), we have

$$(5.9) \quad \begin{aligned} & \langle 1, H_\beta 1 \rangle \\ &= \langle 1_{U^c}, \check{H}^{U^c} 1_{U^c} \rangle + \langle 1_U, H_{U, U} 1_U \rangle + \left\langle 1_{U^c}, P_{U^c, U} \hat{G}^U P_{U, U^c} 1_{U^c} \right\rangle - 2 \langle 1_U, P_{U, U^c} 1_{U^c} \rangle \end{aligned}$$

On the other hand, by (5.8) again, we have

$$(5.10) \quad G_\beta = \begin{pmatrix} I_U & \hat{G}^U P_{U,U^c} \\ 0 & I_{U^c} \end{pmatrix} \begin{pmatrix} \hat{G}^U & 0 \\ 0 & \check{G}^{U^c} \end{pmatrix} \begin{pmatrix} I_U & 0 \\ P_{U^c,U} \hat{G}^U & I_{U^c} \end{pmatrix}$$

therefore, since

$$\begin{pmatrix} I_U & 0 \\ P_{U^c,U} \hat{G}^U & I_{U^c} \end{pmatrix} \begin{pmatrix} \eta_U \\ \eta_{U^c} \end{pmatrix} = \begin{pmatrix} \eta_U \\ \check{\eta} \end{pmatrix},$$

we get,

$$(5.11) \quad \langle \eta, G_\beta \eta \rangle = \langle \eta_U, \hat{G}^U \eta_U \rangle + \langle \check{\eta}, \check{G}^{U^c} \check{\eta} \rangle.$$

Combining (5.9) and (5.11) we have

$$(5.12) \quad \begin{aligned} \langle 1, H_\beta 1 \rangle + \langle \eta, G_\beta \eta \rangle - 2 \langle \eta, 1 \rangle &= \langle 1_{U^c}, \check{H}^{U^c} 1_{U^c} \rangle + \langle \check{\eta}, \check{G}^{U^c} \check{\eta} \rangle - 2 \langle \check{\eta}, 1_{U^c} \rangle \\ &\quad + \langle 1_U, H_{U,U} 1_U \rangle + \langle \hat{\eta}, \hat{G}^U \hat{\eta} \rangle - 2 \langle \hat{\eta}, 1_U \rangle \end{aligned}$$

By (5.8), we also have

$$(5.13) \quad \det H_\beta = \det H_{U,U} \det \check{H}^{U^c}, \quad \mathbb{1}_{H_\beta > 0} = \mathbb{1}_{H_{U,U} > 0} \mathbb{1}_{\check{H}^{U^c} > 0},$$

Combining (5.12) and (5.13), we have,

$$\begin{aligned} &\left(\frac{2}{\pi}\right)^{|V|/2} e^{-\frac{1}{2}\langle 1, H_\beta 1 \rangle - \frac{1}{2}\langle \eta, G_\beta \eta \rangle + \langle \eta, 1 \rangle} \frac{\mathbb{1}_{H_\beta > 0}}{\sqrt{\det H_\beta}} \\ &= \left(\frac{2}{\pi}\right)^{|U|/2} e^{-\frac{1}{2}\langle 1_U, H_{U,U} 1_U \rangle - \frac{1}{2}\langle \hat{\eta}, \hat{G}^U \hat{\eta} \rangle + \langle \hat{\eta}, 1_U \rangle} \frac{\mathbb{1}_{H_{U,U} > 0}}{\det H_{U,U}} \\ &\quad \cdot \left(\frac{2}{\pi}\right)^{|U^c|/2} e^{-\frac{1}{2}\langle 1_{U^c}, \check{H}^{U^c} 1_{U^c} \rangle - \frac{1}{2}\langle \check{\eta}, \check{G}^{U^c} \check{\eta} \rangle + \langle \check{\eta}, 1_{U^c} \rangle} \frac{\mathbb{1}_{\check{H}^{U^c} > 0}}{\sqrt{\det \check{H}^{U^c}}} \end{aligned}$$

We remark that the first term of the right-hand side corresponds to the density of the distribution  $\nu_U^{W_{U,U}, \hat{\eta}}$  and that the second term of the right-hand side is the density of the distribution  $\nu_{U^c}^{\check{W}, \check{\eta}}$ . Integrating on  $d\beta_{U^c}$  on both sides, with  $\beta_U$  fixed, gives

$$\begin{aligned} &\int \left(\frac{2}{\pi}\right)^{|V|/2} e^{-\frac{1}{2}\langle 1, H_\beta 1 \rangle - \frac{1}{2}\langle \eta, G_\beta \eta \rangle + \langle \eta, 1 \rangle} \frac{\mathbb{1}_{H_\beta > 0}}{\sqrt{\det H_\beta}} (d\beta_{U^c}) \\ &= \left(\frac{2}{\pi}\right)^{|U|/2} e^{-\frac{1}{2}\langle 1_U, H_{U,U} 1_U \rangle - \frac{1}{2}\langle \hat{\eta}, \hat{G}^U \hat{\eta} \rangle + \langle \hat{\eta}, 1_U \rangle} \frac{\mathbb{1}_{H_{U,U} > 0}}{\det H_{U,U}} \end{aligned}$$

since  $\int \nu_{U^c}^{\check{W}, \check{\eta}}(d\beta_{U^c}) = 1$  by Lemma 3. Hence, the marginal distribution of  $\beta_U$  is  $\nu_U^{W_{U,U}, \hat{\eta}}$ , proving (i).

Finally, (ii) is a consequence of the conditional probability density formula.  $\square$

**Remark 14.** When  $\eta = 0$ , we have  $\check{\eta} = 0$ , and we only need Theorem A in place of Lemma 3 in the proof.



*Proof of Lemma 5.* We take the same notations as in the proof of Lemma 4. By Lemma 4, the law of  $\beta_{U^c}$ , conditionally on  $\beta_U$ , is  $\nu_{U^c}^{\tilde{W}, \tilde{\eta}}$ . Now

$$\tilde{\psi} = \check{G}^{U^c} \tilde{\eta}.$$

By (5.10), we have

$$\begin{aligned} \langle \lambda, \psi \rangle + \frac{1}{2} \langle \lambda, G_\beta \lambda \rangle &= \langle \lambda, G_\beta \eta \rangle + \frac{1}{2} \langle \lambda, G_\beta \lambda \rangle \\ &= \left( \lambda_U, \lambda_U \hat{G}^U P_{U, U^c} + \lambda_{U^c} \right) \begin{pmatrix} \hat{G}^U & 0 \\ 0 & \check{G}^{U^c} \end{pmatrix} \begin{pmatrix} \eta_U \\ P_{U^c, U} \hat{G}^U \eta_U + \eta_{U^c} \end{pmatrix} \\ &\quad + \frac{1}{2} \left( \lambda_U, \lambda_U \hat{G}^U P_{U, U^c} + \lambda_{U^c} \right) \begin{pmatrix} \hat{G}^U & 0 \\ 0 & \check{G}^{U^c} \end{pmatrix} \begin{pmatrix} \lambda_U \\ P_{U^c, U} \hat{G}^U \lambda_U + \lambda_{U^c} \end{pmatrix} \end{aligned}$$

If we define  $\check{\lambda} = \lambda_{U^c} + P_{U^c, U} \hat{G}^U \lambda_U \in \mathbb{R}_+^{U^c}$ , we have

$$\begin{aligned} \langle \lambda, \psi \rangle + \frac{1}{2} \langle \lambda, G_\beta \lambda \rangle &= \langle \check{\lambda}, \tilde{\psi} \rangle + \frac{1}{2} \langle \check{\lambda}, \check{G}^{U^c} \check{\lambda} \rangle + \langle \lambda_U, \hat{G}^U \eta_U \rangle + \frac{1}{2} \langle \lambda_U, \hat{G}^U \lambda_U \rangle \\ &= \langle \check{\lambda}, \tilde{\psi} \rangle + \frac{1}{2} \langle \check{\lambda}, \check{G}^{U^c} \check{\lambda} \rangle + \langle \lambda_U, \hat{\psi} \rangle + \frac{1}{2} \langle \lambda_U, \hat{G}^U \lambda_U \rangle - \langle 1_{U^c}, \check{\lambda} - \lambda_{U^c} \rangle. \end{aligned}$$

Now, remark that

$$\langle \check{\lambda}, \tilde{\psi} \rangle + \frac{1}{2} \langle \check{\lambda}, \check{G}^{U^c} \check{\lambda} \rangle + \frac{1}{2} \langle \tilde{\eta}, \check{G}^{U^c} \tilde{\eta} \rangle = \frac{1}{2} \langle \check{\lambda} + \tilde{\eta}, \check{G}^{U^c} (\check{\lambda} + \tilde{\eta}) \rangle.$$

We get,

$$\begin{aligned} &\mathbb{E}_{\nu_V^{W, \eta}} \left( e^{-\langle \lambda, \psi \rangle - \frac{1}{2} \langle \lambda, G_\beta \lambda \rangle} \mid \mathcal{F}_U \right) \\ &= e^{-\langle \lambda_U, \hat{\psi} \rangle - \langle \lambda_{U^c}, 1_{U^c} \rangle - \frac{1}{2} \langle \lambda_U, \hat{G}^U \lambda_U \rangle} \mathbb{E}_{\nu_{U^c}^{\tilde{W}, \tilde{\eta}}} \left( e^{-\langle \check{\lambda}, \tilde{\psi} \rangle - \frac{1}{2} \langle \check{\lambda}, \check{G}^{U^c} \check{\lambda} \rangle + \langle 1_{U^c}, \check{\lambda} \rangle} \right) \\ &= e^{-\langle \lambda_U, \hat{\psi} \rangle - \langle \lambda_{U^c}, 1_{U^c} \rangle - \frac{1}{2} \langle \lambda_U, \hat{G}^U \lambda_U \rangle} \mathbb{E}_{\nu_{U^c}^{\tilde{W}, \check{\lambda} + \tilde{\eta}}} (1) \end{aligned}$$

which concludes the proof of the lemma, using that  $\nu_{U^c}^{\tilde{W}, \check{\lambda} + \tilde{\eta}}$  is a probability □

**5.2. Proof of Lemma 2.** Remark that since  $\psi^{(n)}$  is defined for all  $n$  by

$$\begin{cases} (H_\beta \psi^{(n)})_{V_n} = 0, \\ \psi_{V_n^c}^{(n)} = 1, \end{cases}$$

we have  $\psi_{V_n}^{(n)} = ((H_\beta)_{V_n, V_n})^{-1}(\eta^{(n)})$ , where  $\eta^{(n)} = P_{V_n, V_n^c}(1_{V_n^c})$ . Moreover, by Lemma 4 (i), we know that  $\beta_{V_n} \sim \nu_{V_n}^{W, \eta^{(n)}}$ . Using Lemma 2 applied to  $V = V_{n+1}$  and  $U = V_n$ , we have that  $\hat{G}_{V_{n+1}, V_{n+1}}^{(n+1)}$  corresponds to  $G_\beta$  in Lemma 4 and  $\hat{G}_{V_n, V_n}^{(n)}$  to  $\hat{G}^U$ ,  $\eta^{(n+1)}$  to  $\eta$ , and  $\eta^{(n)}$  to  $\hat{\eta}$ . Hence, we get that

$$\begin{aligned} \mathbb{E} \left( e^{-\langle \lambda_{V_{n+1}}, \psi_{V_{n+1}}^{(n+1)} \rangle - \frac{1}{2} \langle \lambda_{V_{n+1}}, \hat{G}^{(n+1)} \lambda_{V_{n+1}} \rangle} \mid \mathcal{F}_n \right) &= e^{-\langle \lambda_{V_n}, \psi_{V_n}^{(n)} \rangle - \langle \lambda_{V_{n+1} \setminus V_n}, 1_{V_{n+1} \setminus V_n} \rangle - \frac{1}{2} \langle \lambda_{V_n}, \hat{G}^{(n)} \lambda_{V_n} \rangle} \\ &= e^{-\langle \lambda_{V_{n+1}}, \psi_{V_{n+1}}^{(n)} \rangle - \frac{1}{2} \langle \lambda_{V_n}, \hat{G}^{(n)} \lambda_{V_n} \rangle} \end{aligned}$$

since  $\psi_{V_{n+1} \setminus V_n}^{(n)} = 1$ . This concludes the proof since  $\psi^{(n)}$  and  $\psi^{(n+1)}$  are both equal 1 on  $V_{n+1}^c$ .

## 6. PASSING TO THE LIMIT : PROOF OF THEOREM 1, PROPOSITION 2, PROPOSITION 3

## 6.1. Representation by sums of paths : proof of Theorem 1 i) and ii).

*Proof of Theorem 1 i).* Recall the definition of  $G^{(n)}$  from section 4.3. By Proposition 6,  $G^{(n)}(i, j)$  is a.s. finite, hence  $\hat{G}^{(n)}(i, j)$  is also a.s. finite since  $\hat{G}^{(n)}(i, j) \leq G^{(n)}(i, j)$  when  $i, j$  are in  $V_n$ , cf (4.7). The sequence  $V_n$  is increasing, hence  $\hat{G}^{(n)}(i, j)$  is an increasing function of  $n$ , to prove Theorem 1 i), it is enough to show that  $\hat{G}(i, j) = \lim_{n \rightarrow \infty} \hat{G}^{(n)}(i, j)$  is a.s. finite.

As  $\hat{G}^{(n)}(i, i)$  converges a.s. to  $\hat{G}(i, i)$ , by dominated convergence, and from (4.7), for any  $h \geq 0$ ,

$$\begin{aligned} \mathbb{P}(\hat{G}(i, i) \leq h) &= \mathbb{P}(\lim_{n \rightarrow \infty} \hat{G}^{(n)}(i, i) \leq h) \\ &= \lim_{n \rightarrow \infty} \mathbb{P}(\hat{G}^{(n)}(i, i) \leq h) \\ &\geq \lim_{n \rightarrow \infty} \mathbb{P}(G^{(n)}(i, i) \leq h) \\ &= \mathbb{P}\left(\frac{1}{2\gamma} \leq h\right), \end{aligned}$$

since by Theorem C,  $G^{(n)}(i, i) \stackrel{\text{law}}{=} \frac{1}{2\gamma}$  where  $\gamma$  is a  $\text{Gamma}(\frac{1}{2}, 1)$  distributed random variable. Therefore,  $\hat{G}(i, i) < \infty$  a.s. For the off diagonal term, as  $H_\beta^{(n)}$  is an M-matrix, in particular,  $(H_\beta^{(n)})|_{V_n}$  is positive definite, we have

$$\hat{G}^{(n)}(i, j) = \langle \delta_i, \hat{G}^{(n)} \delta_j \rangle \leq \sqrt{\langle \delta_i, \hat{G}^{(n)} \delta_i \rangle \langle \delta_j, \hat{G}^{(n)} \delta_j \rangle} = \sqrt{\hat{G}^{(n)}(i, i) \hat{G}^{(n)}(j, j)}$$

therefore,  $\hat{G}(i, j) \leq \sqrt{\hat{G}(i, i) \hat{G}(j, j)}$  and  $\hat{G}(i, j)$  is a.s. finite.  $\square$

*Proof of Theorem 1 ii).* From Proposition 9, we know that  $\psi^{(n)}(k)$  is a positive integrable martingale for all  $k \in V$ . As a positive martingale,  $\psi^{(n)}(k)$  converges a.s. to some non-negative integrable random variable  $\psi(k)$ .

It remains to show that the convergence does not depend on the choice of the exhausting sequence  $(V_n)$ . Assume that  $(\Omega_n)$  is another increasing exhausting sequence, we can similarly construct the martingale  $\phi^{(n)}(k)$  associated to  $\Omega_n$ . As  $(\Omega_n)$  and  $(V_n)$  are exhausting, we can construct a subsequence  $n_k$  such that the sequence  $V_{n_1}, \Omega_{n_2}, V_{n_3}, \dots$  is increasing and thus the sequence  $\psi^{(n_1)}(k), \phi^{(n_2)}(k), \psi^{(n_3)}(k), \dots$  is a martingale for all  $k \in V$ . This martingale converges a.s. and this identifies the limits of  $\psi^{(n)}(k)$  and  $\phi^{(n)}(k)$ .  $\square$

## 6.2. Representation of the VRJP as a mixture on the infinite graphs: proof of iii). Firstly, by Proposition (8) we have

$$G^{(n)}(i, j) = \hat{G}^{(n)}(i, j) + \psi^{(n)}(i) \psi^{(n)}(j) G^{(n)}(\delta_n, \delta_n).$$

From the coupling of Section 4.2, and Theorem 1 i) and ii), we have that a.s.

$$(6.1) \quad \lim_{n \rightarrow \infty} G^{(n)}(i, j) = G(i, j),$$

where  $G(i, j)$  is defined in Theorem 1 iii).

The next corollary of Proposition 7 gives the necessary uniform integrability to extend the representation of the VRJP for finite graphs to infinite graphs.

**Corollary 1.** *For any  $i, j \in V$ , there exists  $n_0 \in \mathbb{N}$ , such that the family of random variables  $\frac{G^{(n)}(i_0, j)}{G^{(n)}(i_0, i)}$ ,  $n \geq n_0$ , is uniformly integrable.*

*Proof.* Choose  $n_0$  such that  $i, j \in V_{n_0}$ , and  $i$  and  $j$  are connected by a path in  $V_{n_0}$ . Denote by  $K$  the distance between  $i$  and  $j$  for the graph distance in  $V_{n_0}$ . Proposition 7 implies that there is  $\eta > 0$  and  $c > 0$ , such that for  $n \geq n_0$ ,

$$\mathbb{E}\left(\left(\frac{G^{(n)}(i_0, j)}{G^{(n)}(i_0, i)}\right)^2\right) \leq c \cdot \mathbb{E}\left(\exp\left(\frac{\eta}{2}\left(\frac{G^{(n)}(i_0, i)}{G^{(n)}(i_0, j)} + \frac{G^{(n)}(i_0, j)}{G^{(n)}(i_0, i)}\right)\right)\right) \leq 2^{K/2}c.$$

The family is uniformly bounded in  $L^2$ , in particular uniformly integrable.  $\square$

Consider now a connected finite subset  $\Lambda \subset V$  containing  $i_0$  and set

$$\partial^+ \Lambda = \{j \in \Lambda^c, \exists i \in \Lambda \text{ such that } i \sim j\}.$$

Let  $T$  be the following stopping time

$$T = \inf\{t \geq 0, Z_t \notin \Lambda\}.$$

By construction, the distribution of  $Z_t$  on  $\mathcal{G}$  up to time  $T$  equals the distribution of  $Z_t$  on  $\mathcal{G}_n$  up to time  $T$ , for all  $n$  such that  $\Lambda \cup \partial^+ \Lambda \subset V_n$ . We denote by

$$\ell_i(T) = \int_0^T \mathbf{1}_{Z_u=i} du,$$

the local time of  $Z$  up to time  $T$ . Using Theorem C and the coupling of Section 4.2, the time-changed VRJP ( $Z_t$ ) on  $\mathcal{G}_n$ , starting at  $i_0$ , is a mixture of Markov jump process with jumping rates from  $i$  to  $j$

$$(6.2) \quad \frac{1}{2} W_{i,j}^{(n)} \frac{G^{(n)}(i_0, j)}{G^{(n)}(i_0, i)}.$$

We denote by

$$\tilde{\beta}_i^{(n)} = \sum_{j \sim i} \frac{1}{2} W_{i,j}^{(n)} \frac{G^{(n)}(i_0, j)}{G^{(n)}(i_0, i)},$$

the holding time at site  $i$ . We denote by  $P_{i_0}^{\text{MJP}}$  the law of the Markov Jump process with jump rates  $\frac{1}{2} W_{i,j}$  starting from  $i_0$ . By simple computation, the Radon-Nykodim derivative of the law of  $(Z_t)_{t \leq T}$  under the Markov jump process with jump rates (6.2) and under  $P_{i_0}^{\text{MJP}}$  is

$$e^{-\sum_{i \in \Lambda} \ell_i(T) (\tilde{\beta}_i^{(n)} - \frac{1}{2} W_i)} \frac{G^{(n)}(i_0, Z_T)}{G^{(n)}(i_0, i_0)}$$

where as usual  $W_i = \sum_{j \sim i} W_{i,j}$ . It implies that for all positive bounded test function  $F$ .

$$\begin{aligned} & \mathbb{E}_{i_0}^{\text{VRJP}, \mathcal{G}_n} (F((Z_t)_{t \leq T})) \\ &= \int \sum_{j \in \partial^+ \Lambda} E_{i_0}^{\text{MJP}} \left( \mathbf{1}_{Z_T=j} F((Z_t)_{t \leq T}) e^{-\sum_{i \in \Lambda} \ell_i(T) (\tilde{\beta}_i^{(n)} - \frac{1}{2} W_i)} \frac{G^{(n)}(i_0, j)}{G^{(n)}(i_0, i_0)} \right) \nu_V^W(d\beta, d\gamma) \\ &= \sum_{j \in \partial^+ \Lambda} E_{i_0}^{\text{MJP}} \left( \mathbf{1}_{Z_T=j} F((Z_t)_{t \leq T}) e^{\sum_{i \in \Lambda} \frac{1}{2} \ell_i(T) W_i} \int e^{-\sum_{i \in \Lambda} \ell_i(T) \tilde{\beta}_i^{(n)}} \frac{G^{(n)}(i_0, j)}{G^{(n)}(i_0, i_0)} \nu_V^W(d\beta, d\gamma) \right) \end{aligned}$$

where  $\nu_V^W(d\beta, d\gamma)$  is the joint law of  $\beta, \gamma$ , defined in Theorem 1 and Section 4.2. From (6.1), we have a.s.

$$\lim_{n \rightarrow \infty} \tilde{\beta}_i^{(n)} = \tilde{\beta}_i = \sum_{j \sim i} \frac{1}{2} W_{i,j} \frac{G(i_0, j)}{G(i_0, i)}$$

Letting  $n$  go to infinity, using the uniform integrability of  $\frac{G^{(n)}(i_0, j)}{G^{(n)}(i_0, i_0)}$ , Corollary 1, we get that

$$\begin{aligned} & \mathbb{E}_{i_0}^{\text{VRJP}, \mathcal{G}} (F((Z_t)_{t \leq T})) \\ &= \sum_{j \in \partial^+ \Lambda} E_{i_0}^{\text{MJP}} \left( \mathbf{1}_{Z_T=j} F((Z_t)_{t \leq T}) e^{\sum_{i \in \Lambda} \frac{1}{2} \ell_i(T) W_i} \int e^{-\sum_{i \in \Lambda} \ell_i(T) \tilde{\beta}_i} \frac{G(i_0, j)}{G(i_0, i_0)} \nu_V^W(d\beta, d\gamma) \right) \\ &= \int E_{i_0}^{\beta, \gamma, i_0} (F((Z_t)_{t \leq T})) \nu_V^W(d\beta, d\gamma) \end{aligned}$$

where  $E_{i_0}^{\beta, \gamma, i_0}$  is the expectation associated with the probability  $P_{i_0}^{\beta, \gamma, i_0}$  defined in Theorem 1. This concludes the proof of iii) of that Theorem.

### 6.3. Proof of Proposition 2, and iv) of Theorem 1.

*Proof of Proposition 2.* Recall (3.6) and (3.7). As  $n \mapsto \hat{G}^{(n)}(i, j)$  is increasing, we have

$$\hat{G}(i, j) = \sum_{\sigma \in \mathcal{P}_{i,j}^V} \frac{W_\sigma}{2\beta_\sigma}.$$

By arguments similar to (3.10), we have

$$\frac{\hat{G}(i_0, i)}{\hat{G}(i_0, i_0)} = \sum_{\sigma \in \mathcal{P}_{i,i_0}^V} \frac{W_\sigma}{(2\beta)_\sigma^-}.$$

Therefore, if we denote  $\{(\tilde{Z}_n) \sim \sigma\} = \{\tilde{Z}_0 = \sigma_0, \dots, \tilde{Z}_m = \sigma_m\}$  with  $m = |\sigma|$ , then for  $i \neq i_0$

$$\begin{aligned} h(i) &:= P_i^{\beta, \gamma, i_0}(\tau_{i_0} < \infty) = \sum_{\sigma \in \mathcal{P}_{i,i_0}^V} P_i^{\beta, \gamma, i_0}((\tilde{Z}_n) \sim \sigma) \\ (6.3) \quad &= \sum_{\sigma \in \mathcal{P}_{i,i_0}^V} \frac{W_\sigma}{(2\beta)_\sigma^-} \frac{G(i_0, i_0)}{G(i_0, i)} = \frac{\hat{G}(i_0, i)}{\hat{G}(i_0, i_0)} \cdot \frac{G(i_0, i_0)}{G(i_0, i)}. \end{aligned}$$

It follows from  $G(i, j) = \hat{G}(i, j) + \frac{1}{2\gamma} \psi(i) \psi(j)$  that, for  $i \neq i_0$ ,

$$\begin{aligned} P_i^{\beta, \gamma, i_0}(\tau_{i_0} = \infty) &= 1 - h(i) \\ &= \frac{\psi(i_0)}{2\gamma} \frac{\hat{G}(i_0, i_0) \psi(i) - \hat{G}(i_0, i) \psi(i_0)}{\hat{G}(i_0, i_0) G(i_0, i)}. \end{aligned}$$

Therefore,

$$\begin{aligned} P_{i_0}^{\beta, \gamma, i_0}(\tau_{i_0}^+ = \infty) &= \sum_{j \sim i_0} \frac{W_{i_0, j} G(i_0, j)}{2\tilde{\beta}_{i_0} G(i_0, i_0)} P_j^{\beta, \gamma, i_0}(\tau_{i_0} = \infty) \\ &= \sum_{j \sim i_0} \frac{\psi(i_0) W_{i_0, j}}{4\gamma \tilde{\beta}_{i_0}} \frac{\hat{G}(i_0, i_0) \psi(j) - \hat{G}(i_0, j) \psi(i_0)}{\hat{G}(i_0, i_0) G(i_0, i_0)}. \end{aligned}$$

Taking the limit  $n \rightarrow \infty$  in (4.4), we have  $H_\beta \hat{G}(i_0, \cdot) = \mathbf{1}_{i_0}$ . By (iii) of Theorem 2 (proved in section 6.5), we have  $H_\beta \psi(\cdot) = 0$ , therefore,

$$\sum_{j \sim i_0} W_{i_0, j} [\psi(j) \hat{G}(i_0, i_0) - \psi(i_0) \hat{G}(i_0, j)] = \psi(i_0),$$

hence

$$P_{i_0}^{\beta, \gamma, i_0}(\tau_{i_0}^+ = \infty) = \frac{\psi(i_0)^2}{4\gamma \tilde{\beta}_{i_0} \hat{G}(i_0, i_0) G(i_0, i_0)}.$$

□

**Remark 15.** By maximum principle we can check directly that  $\hat{G}(i, i)\psi(j) - \hat{G}(i, j)\psi(i)$  is nonnegative. Indeed, let

$$h_1^{(n)}(j) := \frac{\psi^{(n)}(j)}{\psi^{(n)}(i)} \hat{G}^{(n)}(i, i), \quad h_2^{(n)}(j) := \hat{G}^{(n)}(i, j).$$

We have  $h_1^{(n)}(i) = h_2^{(n)}(i)$ ,  $h_1^{(n)}(\delta_n) \geq h_2^{(n)}(\delta_n)$  and  $H_\beta^{(n)} h^{(n)} = 0$  outside  $\{i, \delta_n\}$  for  $\cdot \in \{1, 2\}$ , which means that  $h_1^{(n)}, h_2^{(n)}$  are  $H_\beta^{(n)}$ -harmonic, and  $h_1^{(n)} \geq h_2^{(n)}$  on the boundary. This implies that  $h_1^{(n)} \geq h_2^{(n)}$ , and the inequality by letting  $n$  go to  $\infty$ .

*Proof of Theorem 1, (iv).* From Proposition 2, we see that  $P_{i_0}^{\beta, \gamma, i_0}(\tau_{i_0}^+ = \infty) > 0$  if and only if  $\psi(i_0) > 0$ . Since the Markov jump process  $P_{i_0}^{\beta, \gamma, i_0}$  is irreducible ( $\mathcal{G}$  is connected), it implies (iv). □

#### 6.4. Ergodicity and the 0-1 law : proof of Proposition 3 and 5.

*Proof of Proposition 3.* From the expression of the Laplace transform of  $\beta$ , c.f. Proposition 1, we see that  $\beta$  is stationary for the action of  $\mathcal{A}$ . By 1-dependence, c.f. Proposition 1, it is also ergodic. Indeed, assume that  $(\tau_n) \in \mathcal{A}^{\mathbb{N}}$  is a sequence of automorphisms such that  $d_{\mathcal{G}}(i_0, \tau_n(i_0)) \rightarrow \infty$  for some vertex  $i_0$ . We prove that  $(\tau_n)$  is mixing in the sense that for all  $A, B \in \sigma(\beta_i, i \in V)$

$$\lim_{n \rightarrow \infty} \mathbb{P}(\tau_n^{-1}(B) \cap A) = \mathbb{P}(A)\mathbb{P}(B),$$

which clearly implies ergodicity. Assume that  $V_1 \subset V$  is finite and that  $A, B \in \sigma(\beta_j, j \in V_1)$ . By 1-dependence,  $\tau_n^{-1}(B)$  is independent of  $A$  for  $n$  large enough. Hence, the property is true for  $\sigma(\beta_j, j \in V_1)$ -measurable sets. It can be extended by a monotone class argument.

Since  $\psi$  and  $\hat{G}$  are constructed as almost sure limit from functions of  $\beta$ , and since the limit does not depend on the choice of the approximating sequence, then  $\psi$  and  $\hat{G}$  are stationary and ergodic for the action of  $\mathcal{A}$ .

The event  $\{\psi(i) = 0, \forall i \in V\}$  is clearly invariant by  $\mathcal{A}$ , hence has probability 0 or 1. Together with (iv) of Theorem 1 it concludes the proof of the proposition. □

*Proof of Proposition 5.* It works exactly in the same way. □

#### 6.5. Proof of Theorem 2: relation with spectral properties of the random schrödinger operator.

*Proof of Theorem 2 (i).* As  $H_\beta^{(n)} > 0$  a.s., we have that  $(H_\beta)_{|V_n \times V_n} > 0$  and passing to the limit, we get  $H_\beta \geq 0$ . Hence,  $\sigma(H_\beta) \subset [0, +\infty)$ . □

*Proof of Theorem 2 (ii).* As  $-\varepsilon$  is strictly outside the spectrum of  $H_\beta$ , the equation  $(H_\beta + \varepsilon)\hat{G}_\varepsilon = \text{Id}$  has unique finite solution, we can verify that  $\sum_{\sigma \in \mathcal{P}_{i,j}^V} \frac{W_\sigma}{(2\beta + \varepsilon)_\sigma}$  is a solution to this equation. Now by Theorem 1 (i), we have

$$\hat{G}_\varepsilon(i, j) = \sum_{\sigma \in \mathcal{P}_{i,j}^V} \frac{W_\sigma}{(2\beta + \varepsilon)_\sigma} \leq \hat{G}(i, j) < \infty.$$

Therefore, as  $\sum_{\sigma \in \mathcal{P}_{i,j}} \frac{W_\sigma}{(2\beta + \varepsilon)_\sigma}$  is increasing as  $\varepsilon \rightarrow 0$ , it converges a.s. to  $\hat{G}(i, j)$ . Moreover, it can be verified by direct computation on sums of path that  $H_\beta \hat{G} = \text{Id}$ .  $\square$

*Proof of Theorem 2 (iii).* We have, for all  $i \in V_n$ ,

$$\psi^{(n)}(i) = \sum_{j \sim i} \frac{W_{i,j}}{2\beta_i} \psi^{(n)}(j).$$

As  $\psi^{(n)}(i)$  converges a.s. to  $\psi$ , the above equality holds in the limit, i.e., for all  $i \in V$ ,

$$\psi(i) = \sum_{j \sim i} \frac{W_{i,j}}{2\beta_i} \psi(j),$$

this exactly means  $(H_\beta \psi)(i) = 0$ .  $\square$

*Proof of Theorem 2 (iv).* By Fatou's Lemma, the limit  $\psi(i)$  satisfies  $\mathbb{E}(\psi(i)) \leq 1$ . By Markov inequality

$$\mathbb{P}(\psi(i) \geq C\|i\|^p) \leq \frac{1}{C\|i\|^p}.$$

Let  $\partial B(0, n)$  be the sphere of radius  $n$ , i.e.  $\partial B(0, n) = \{j \in \mathbb{Z}^d, d(0, j) = n\}$ . When  $p > d$ .

$$\begin{aligned} \sum_{i \in \partial B(0, n)} \mathbb{P}(\psi(i) \geq C\|i\|^p) &\leq \sum_{i \in \partial B(0, n)} \frac{1}{C\|i\|^p} \\ &\leq C' \sum_n \frac{n^{d-1}}{n^p} < \infty \end{aligned}$$

for some constant  $C' > 0$ . By Borel-Cantelli lemma, a.s. only a finite number of  $i$  satisfies  $\psi(i) \geq C\|i\|^p$ .  $\square$

## 7. $h$ -TRANSFORMS

**Corollary 2.** (i) *The quenched process  $(Z_t)$  on  $\mathcal{G}$ , conditionally on  $\{\tau_{i_0}^+ < \infty\}$ , up to its first return time to  $i_0$ , is equal in law to the Markov jump process of jump rate from  $i$  to  $j$*

$$\begin{cases} \frac{1}{2} W_{i,j} \frac{\hat{G}(i_0, j)}{\hat{G}(i_0, i)} & i \neq i_0 \\ \tilde{\beta}_{i_0} \frac{W_{i_0, j} \hat{G}(i_0, j)}{\sum_{k \sim i_0} W_{i_0, k} \hat{G}(i_0, k)} & i = i_0, j \sim i_0 \end{cases}$$

where as usual  $\tilde{\beta}_{i_0} = \sum_{j \sim i_0} \frac{1}{2} W_{i_0, j} \frac{G(i_0, j)}{G(i_0, i_0)}$ . Its law is denoted  $\hat{P}_{i_0}^{\beta, i_0}$  in the sequel.



(ii) The annealed process  $(Z_t)$  conditionally on  $\{\tau_{i_0}^+ < \infty\}$ , up to its first return time to  $i_0$ , is given by the following mixture :

$$\mathbb{P}_{i_0}^{VRJP}((Z_t)_{t \leq \tilde{\tau}_{i_0}^+} \in \cdot \mid \tau_{i_0}^+ < \infty) = \int \hat{P}_{i_0}^{\beta, \gamma, i_0}((Z_t)_{t \leq \tilde{\tau}_{i_0}^+} \in \cdot) \frac{P_{i_0}^{\beta, \gamma, i_0}(\tau_{i_0}^+ < \infty)}{\mathbb{P}_{i_0}^{VRJP}(\tau_{i_0}^+ < \infty)} d\mu_{i_0}^{\mathcal{G}, W}(\beta, \gamma),$$

where  $\tilde{\tau}_{i_0}^+ = \inf\{t \geq 0, Z_t = i_0, \exists 0 < s < t \text{ s.t. } Z_s \neq i_0\}$  is the first return time to  $i_0$  of the continuous process  $(Z_t)$ .

(iii) The quenched process  $(Z_t)$ , conditionally on the event  $\{\tau_{i_0}^+ = \infty\}$ , is the Markov jump process with jump rate from  $i$  to  $j$

$$\begin{cases} \frac{1}{2} W_{i,j} \frac{\check{G}(i_0, j)}{\check{G}(i_0, i)} & i \neq i_0, j \neq i_0 \\ \tilde{\beta}_{i_0} \frac{W_{i_0, j} \check{G}(i_0, j)}{\sum_{k \sim i_0} W_{i_0, k} \check{G}(i_0, k)} & i = i_0, j \sim i_0 \\ 0 & i \sim i_0, j = i_0 \end{cases}$$

where  $\check{G}(i, j) = \hat{G}(i, i)\psi(j) - \hat{G}(i, j)\psi(i)$ . Denote by  $\check{P}_{i_0}^{\beta, \gamma, i_0}$  the law of this Markov process.

(iv) The annealed process  $(Z_t)$  conditionally on the event  $\{\tau_{i_0}^+ = \infty\}$ , is a mixture of Markov jump process with mixing law

$$\mathbb{P}_{i_0}^{VRJP}(\cdot \mid \tau_{i_0}^+ = \infty) = \int \check{P}_{i_0}^{\beta, \gamma, i_0}(\cdot) \frac{P_{i_0}^{\beta, \gamma, i_0}(\tau_{i_0}^+ = \infty) d\mu_{i_0}^{\mathcal{G}, W}(\beta, \gamma)}{\mathbb{P}_{i_0}^{VRJP}(\tau_{i_0}^+ = \infty)}.$$

**Remark 16.** Note that in the case (i), the conditional jump rates do not depend on  $\gamma$ .

*Proof of Corollary 2.* (i) Recall from (6.3) that for  $i \neq i_0$

$$h(i) = P_i^{\beta, \gamma, i_0}(\tau_{i_0} < \infty) = \frac{\hat{G}(i_0, i)G(i_0, i_0)}{\hat{G}(i_0, i_0)G(i_0, i)}.$$

and  $h(i_0) = 1$ . For  $i \neq i_0$ , we have

$$P^{\beta, \gamma, i_0}(X_{t+dt} = j \mid X_t = i, t \leq \tau_{i_0}^+ < \infty) \sim \frac{h(j)}{h(i)} P^{\beta, \gamma, i_0}(X_{t+dt} = j \mid X_t = i)$$

Hence, the jumping rate of  $P^{\beta, \gamma, i_0}(\cdot \mid \tau_{i_0}^+ < \infty)$ , up to time  $\tau_{i_0}^+$ , from  $i$  to  $j$  is

$$\frac{1}{2} W_{i,j} \frac{G(i_0, j)}{G(i_0, i)} \frac{h(j)}{h(i)} = \frac{1}{2} W_{i,j} \frac{\hat{G}(i_0, j)}{\hat{G}(i_0, i)}.$$

The jumping rate of  $P^{\beta, \gamma, i_0}(\cdot \mid \tau_{i_0}^+ < \infty)$ , up to time  $\tau_{i_0}^+$ , from  $i_0$  to  $j$  is given by

$$\frac{1}{2} W_{i_0, j} \frac{G(i_0, j)}{G(i_0, i_0)} \frac{h(j)}{P_{i_0}^{\beta, \gamma, i_0}(\tau_{i_0}^+ < \infty)} = \tilde{\beta}_{i_0} \frac{W_{i_0, j} \hat{G}(i_0, j)}{\sum_{k \sim i_0} W_{i_0, k} \hat{G}(i_0, k)}.$$

where  $\tilde{\beta}_{i_0} = \sum_{l \sim i_0} \frac{1}{2} W_{i_0, l} \frac{G(i_0, l)}{G(i_0, i_0)}$ .

- (ii) Recall that  $\mathbb{P}_{i_0}^{\text{VRJP}}$  denotes the probability of VRJP starting at  $i_0$ , and  $\nu_V^W$  the joint law of  $(\beta, \gamma)$ .

$$\begin{aligned} & \mathbb{P}_{i_0}^{\text{VRJP}}((Z_t)_{t \leq \tau_{i_0}^+} \in \cdot \mid \tau_{i_0}^+ < \infty) \\ &= \int P_{i_0}^{\beta, \gamma, i_0}((Z_t)_{t \leq \tau_{i_0}^+} \in \cdot \mid \tau_{i_0}^+ < \infty) \frac{P_{i_0}^{\beta, \gamma, i_0}(\tau_{i_0}^+ < \infty)}{\mathbb{P}_{i_0}^{\text{VRJP}}(\tau_{i_0}^+ < \infty)} \nu_V^W(d\beta, d\gamma) \\ &= \int \hat{P}_{i_0}^{\beta, i_0}((Z_t)_{t \leq \tau_{i_0}^+} \in \cdot) \frac{P_{i_0}^{\beta, \gamma, i_0}(\tau_{i_0}^+ < \infty)}{\mathbb{P}_{i_0}^{\text{VRJP}}(\tau_{i_0}^+ < \infty)} \nu_V^W(d\beta, d\gamma). \end{aligned}$$

- (iii) Similarly to (i), for  $i \neq i_0$ , we have

$$P^{\beta, \gamma, i_0}(X_{t+dt} = j \mid X_t = i, \tau_{i_0}^+ = \infty) \sim \frac{1 - h(j)}{1 - h(i)} P^{\beta, \gamma, i_0}(X_{t+dt} = j \mid X_t = i)$$

Hence, the jumping rate of  $P^{\beta, \gamma, i_0}(\cdot \mid \tau_{i_0}^+ = \infty)$ , from  $i$  to  $j$  is

$$\frac{1}{2} W_{i,j} \frac{G(i_0, j)}{G(i_0, i)} \frac{1 - h(j)}{1 - h(i)} = \frac{1}{2} W_{i,j} \frac{\hat{G}(i_0, i_0) \psi(j) - \hat{G}(i_0, j) \psi(i_0)}{\hat{G}(i_0, i_0) \psi(i) - \hat{G}(i_0, i) \psi(i_0)} = \frac{1}{2} W_{i,j} \frac{\check{G}(i_0, j)}{\check{G}(i_0, i)}.$$

The jumping rate of  $P^{\beta, \gamma, i_0}(\cdot \mid \tau_{i_0}^+ = \infty)$ , from  $i_0$  to  $j$  is given by

$$\frac{1}{2} W_{i_0, j} \frac{G(i_0, j)}{G(i_0, i_0)} \frac{1 - h(j)}{P^{\beta, \gamma, i_0}(\tau_{i_0}^+ = \infty)} = \tilde{\beta}_{i_0} \frac{W_{i_0, j} \check{G}(i_0, j)}{\sum_{k \sim i_0} W_{i_0, k} \check{G}(i_0, k)}.$$

where  $\tilde{\beta}_{i_0} = \sum_{l \sim i_0} \frac{1}{2} W_{i_0, l} \frac{G(i_0, l)}{G(i_0, i_0)}$ .

- (iv) follows easily from (iii) in the same way as in (ii). □

## 8. PROOF OF RECURRENCE OF 2-DIMENSIONAL ERRW : THEOREM 5

Consider the cubical graph  $\mathcal{G} = (\mathbb{Z}^2, E)$  with constant edge weight  $a_e = a > 0$ . From Section 2.4, we know that the ERRW on  $\mathbb{Z}^2$  is a mixture of reversible Markov chains with conductances  $(x_{i,j})$  given in (2.5). We will use [14] to prove that there exists  $c(a) > 0$  and  $\xi(a) > 0$  depending only on  $a$  such that

$$\mathbb{E} \left( \left( \frac{x_\ell}{x_0} \right)^{\frac{1}{4}} \right) \leq c(a) |\ell|_\infty^{-\xi(a)},$$

where  $x_i = \sum_{j \sim i} x_{i,j}$ . This last estimate follows from Theorem 2.8 of [14] (it can also be deduced from Lemma 2.5 of [16]) which gives a similar estimate on finite boxes. In Theorem 2.8 of [14], the estimate is stated for a periodic torus, but it is clear in the proof that the only necessary point is that the finite graph is invariant by the reflection exchanging 0 and  $\ell$ . For this reason we choose the approximating sequence  $V_n = B(\frac{\ell}{2}, n) \cap \mathbb{Z}^2$ , where  $B(\frac{\ell}{2}, n)$  is the ball with center  $\ell/2$  and radius  $n$ . Consider as in Section 4.2 the graph

$$\mathcal{G}_n = (\tilde{V}_n = V_n \cup \{\delta_n\}, E_n),$$

and the associated weights  $(a_e^{(n)})_{e \in E_n}$  obtained by restriction of  $(\mathcal{G}, (a_e)_{e \in E})$  to  $V_n$  with wired boundary condition. Clearly, central symmetry with respect to  $\frac{\ell}{2}$  (mapping  $\delta_n$  to itself) leaves invariant  $(\mathcal{G}^{(n)}, a^{(n)})$  and exchange 0 and  $\ell$ .

Following the discussion of Section 2.4, we consider as in (2.5) for  $i \sim j$  in  $\mathbb{Z}^2$

$$x_{i,j} = W_{i,j}G(0,i)G(0,j),$$

where  $(W, \beta, \gamma)$  are distributed according to  $\tilde{\nu}_V^a(dW, d\beta, d\gamma)$ . With the coupling defined in Section 4.2, we define for  $i \sim j$ ,  $i, j$  in  $\tilde{V}_n$ ,

$$x_{i,j}^{(n)} = W_{i,j}^{(n)}G^{(n)}(0,i)G^{(n)}(0,j).$$

where  $W^{(n)}$  is obtained by restriction with wired boundary condition from  $W$ . By additivity of Gamma random variables,  $(W_e^{(n)})_{e \in E_n}$  are independent Gamma random variables with parameters  $(a_e^{(n)})_{e \in E_n}$ . Hence, the ERRW on  $V_n$ , with initial weights  $a^{(n)}$ , starting from 0, is a mixture of reversible Markov chains with conductances  $(x_e^{(n)})_{e \in E_n}$ .

From Theorem 1, with the coupling defined in Section 4.2, we have that for all  $i, j \in \mathbb{Z}^2$ ,  $i \sim j$ , a.s.

$$(8.1) \quad \lim_{n \rightarrow \infty} x_{i,j}^{(n)} = x_{i,j}.$$

The proof of Theorem 2.8 of [14], can be readily adapted to prove the following estimate.

**Lemma 6.** *In the case of constant weights  $a_e = a$ , there exists  $c(a) > 0$  and  $\xi(a) > 0$  only depending on  $a$  such that for  $n$  large enough*

$$\mathbb{E} \left( \left( \frac{x_\ell^{(n)}}{x_0^{(n)}} \right)^{\frac{1}{4}} \right) \leq c(a) |\ell|_\infty^{-\xi(a)}.$$

By (8.1) and Fatou's lemma,

$$(8.2) \quad \mathbb{E} \left( \left( \frac{x_\ell}{x_0} \right)^{\frac{1}{4}} \right) \leq c(a) |\ell|_\infty^{-\xi(a)}.$$

We now deduce recurrence of the ERRW from that estimate and from Theorem 1 and Proposition 5. We have, for  $\ell \neq 0$ ,

$$x_\ell = \sum_{j \sim \ell} W_{\ell,j}G(0,\ell)G(0,j) = 2\beta_\ell G(0,\ell)^2 \geq \frac{\beta_\ell}{2\gamma^2} \psi(0)^2 \psi(\ell)^2.$$

Similarly,

$$x_0 = \sum_{j \sim 0} W_{0,j}G(0,0)G(0,j) = G(0,0)(2\beta_0 G(0,0) - 1).$$

Hence,

$$(8.3) \quad \frac{x_\ell}{x_0} \geq \frac{\psi(0)^2}{2\gamma^2 G(0,0)(2\beta_0 G(0,0) - 1)} \beta_\ell \psi_\ell^2.$$

Assume the ERRW is transient. By Proposition 5 it implies that, a.s.,  $\psi(i) > 0$  for all  $i$ . Moreover  $\beta_\ell \psi_\ell^2$  is stationary with respect to translations. Let  $\epsilon > 0$ , we have by (8.2)

$$(8.4) \quad \mathbb{P} \left( \frac{x_\ell}{x_0} \geq \epsilon \right) \leq \frac{1}{\epsilon^4} c(a) |\ell|_\infty^{-\xi(a)}.$$

Consider  $\eta > 0$  such that

$$\mathbb{P} \left( \frac{\psi(0)^2}{2\gamma^2 G(0,0)(2\beta_0 G(0,0) - 1)} \leq \eta \right) \leq \frac{1}{2}.$$

We have by (8.3)

$$\begin{aligned} \mathbb{P}\left(\frac{x_\ell}{x_0} \geq \epsilon\right) &\geq \mathbb{P}\left(\frac{\psi(0)^2}{2\gamma^2 G(0,0)(2\beta_0 G(0,0) - 1)} > \eta, \quad \beta_l \psi(l)^2 > \frac{\epsilon}{\eta}\right) \\ &= 1 - \mathbb{P}\left(\left\{\frac{\psi(0)^2}{2\gamma^2 G(0,0)(2\beta_0 G(0,0) - 1)} \leq \eta\right\} \cup \left\{\beta_l \psi(l)^2 \leq \frac{\epsilon}{\eta}\right\}\right) \\ &\geq \frac{1}{2} - \mathbb{P}\left(\beta_l \psi(l)^2 \leq \frac{\epsilon}{\eta}\right). \end{aligned}$$

Since  $\beta_l \psi(l)^2$  is stationary and by (8.4), we get

$$\mathbb{P}\left(\beta_0 \psi(0)^2 \leq \frac{\epsilon}{\eta}\right) = \mathbb{P}\left(\beta_l \psi(l)^2 \leq \frac{\epsilon}{\eta}\right) \geq \frac{1}{2} - \frac{1}{\epsilon^{\frac{1}{4}}} c(a) |\ell|_\infty^{-\xi(a)}.$$

By sending  $\ell$  to infinity, we get  $\mathbb{P}\left(\beta_0 \psi(0)^2 \leq \frac{\epsilon}{\eta}\right) \geq \frac{1}{2}$ . This is incompatible with  $\psi(0) > 0$  a.s., hence with transience of ERRW.

## 9. PROOF OF FUNCTIONAL CENTRAL LIMIT THEOREMS FOR THE VRJP AND THE ERRW : THEOREM 3 AND 4

*Proof of Theorem 3 and Theorem 4.* Let us start by the VRJP with constant weights  $W_{i,j} = W$ . Assume that the VRJP is transient.

Denote by  $(X_n)_{n \in \mathbb{N}}$  the canonical process on  $(\mathbb{Z}^d)^\mathbb{N}$ . Given the environment  $\beta, \gamma$ , let us define  $\tilde{P}^\psi$  to be the law of the reversible Markov chain with conductances  $W_{i,j} \psi(i) \psi(j)$ , i.e. with transition probabilities

$$\tilde{P}^\psi(X_{n+1} = j | X_n = i) = \frac{W_{i,j} \psi(j)}{\sum_{l \sim i} W_{i,l} \psi(l)}.$$

Denote by  $\tilde{P}^{\beta, \gamma, 0}$  the law of the underlying discrete time process associated with the Markov Jump process  $P^{\beta, \gamma, 0}$ , so that for  $i \sim j$

$$\tilde{P}^{\beta, \gamma, 0}(X_{n+1} = j | X_n = i) = \frac{W_{i,j} G(0, j)}{\sum_{l \sim i} W_{i,l} G(0, l)}.$$

As  $\psi$  is a generalized eigenfunction of  $H_\beta$ , for any  $i \in V$

$$\sum_{j \sim i} \frac{1}{2} W_{i,j} \frac{\psi(j)}{\psi(i)}.$$

It then follows by Proposition 6 that, for  $i \neq 0$

$$\begin{aligned} h^\psi(i) &:= \tilde{P}_i^\psi(\tau_0 < \infty) = \sum_{\sigma \in \tilde{\mathcal{P}}_{i,0}^V} \tilde{P}_i^\psi(Z_n \sim \sigma) \\ &= \sum_{\sigma \in \tilde{\mathcal{P}}_{i,0}^V} \frac{W_\sigma}{(2\beta)_\sigma} \frac{\psi(0)}{\psi(i)} = \frac{\hat{G}(0, i) \psi(0)}{\hat{G}(0, 0) \psi(i)}. \end{aligned}$$

(recall that  $\bar{\mathcal{P}}_{i,0}^V$  is also defined in Proposition 6.) Consider the Markov chain  $\tilde{P}_0^\psi(\cdot | \tau_0^+ = \infty)$  (Doob's  $(1 - h^\psi)$ -transform). By similar computation as in the proof of Proposition 2, we

have that the transition probability of  $\tilde{P}_0^\psi(\cdot | \tau_0^+ = \infty)$  from  $i$  to  $j$  is, for  $j \neq 0$ ,

$$\frac{W_{i,j}\psi(j)(1-h^\psi(j))}{\sum_{l \sim i} W_{i,l}\psi(l)(1-h^\psi(l))} = \frac{W_{i,j}\check{G}(0,j)}{\sum_{l \sim i} W_{i,l}\check{G}(0,l)}$$

and 0 when  $j = 0$ . Therefore, we see that the transition probabilities of  $\tilde{P}_0^\psi(\cdot | \tau_0^+ = \infty)$  are the same as those of  $\tilde{P}_0^{\beta,\gamma,0}(\cdot | \tau_0^+ = \infty)$ , cf iii) of Proposition 2. Moreover, if we denote

$$\xi_0 = \sup\{n; X_n = 0\}$$

then by strong Markov property

$$\tilde{P}_0^\psi(X_n \in \cdot | \tau_0^+ = \infty) = \tilde{P}_0^\psi((X \circ \theta_{\xi_0})_n \in \cdot)$$

$$\tilde{P}_0^{\beta,\gamma,0}(X_n \in \cdot | \tau_0^+ = \infty) = \tilde{P}_0^{\beta,\gamma,0}((X \circ \theta_{\xi_0})_n \in \cdot)$$

where  $\theta_n$  is the shift in time by  $n$ . It follows that  $(X \circ \theta_{\xi_0})_n$  has the same law under  $\tilde{P}_0^\psi$  and under  $\tilde{P}_0^{\beta,\gamma,0}$ .

Remark also, from Proposition 3, that  $W_{i,j}\psi(i)\psi(j)$  are stationary and ergodic conductances. We can thus apply Theorem 4.5 and Theorem 4.6 of [6]. In order to have a functional central limit theorem we need to show that, cf Theorem 4.5 of [6],

$$(9.1) \quad \mathbb{E}(W_{i,j}\psi(i)\psi(j)) < \infty.$$

In order to show that it has non-degenerate asymptotic covariance we need to show that, cf Theorem 4.6 and identity (4.20) of [6],

$$(9.2) \quad \mathbb{E}\left(\frac{1}{W_{i,j}\psi(i)\psi(j)}\right) < \infty.$$

By invariance of the law of the conductances by symmetries of  $\mathbb{Z}^d$ , we know that the limit diffusion matrix is of the form  $\sigma^2 \text{Id}$ .

The same reasoning works in the case of the ERRW with constant weights  $a_{i,j} = a$  : in this case  $(W_{i,j})$  are i.i.d., but as shown in Proposition 5,  $W_{i,j}\psi(i)\psi(j)$  is also stationary and ergodic under  $\tilde{\nu}_V^a(dW, d\beta)$ .

Estimates (9.1) and (9.2) are provided by [10] in the VRJP case, and by [8] in the ERRW case. This is summarized in the following lemma.

**Lemma 7.** (i) (VRJP case) Consider the VRJP on  $\mathbb{Z}^d$ , for  $d \geq 3$ , with constant weights  $W_{ij} = W$ . There exists  $0 < \lambda_2 < \infty$  such that for  $W > \lambda_2$ , the VRJP is transient and such that (9.1), (9.2) are true under  $\nu_V^W(d\beta)$ .

(ii) (ERRW case) Consider the ERRW on  $\mathbb{Z}^d$ , for  $d \geq 3$ , with constant weights  $a_{ij} = a$ . There exists  $0 < \tilde{\lambda}_2 < \infty$  such that for  $a > \tilde{\lambda}_2$ , the ERRW is transient and (9.1), (9.2) are true under  $\tilde{\nu}_V^a(dW, d\beta)$ .

The proof of that lemma is given below. We first apply it to prove the functional central limit theorem.

Consider the VRJP case and assume that the condition of the lemma is satisfied. Define

$$X_t^{(n)} = \frac{X_{[nt]}}{\sqrt{n}}.$$

From [6], we know that there exists  $0 < \sigma^2 < \infty$  such that for all bounded Liptchitz function  $F$  for the Skorokhod topology, for all  $\epsilon > 0$ , for all  $0 < T < \infty$ ,

$$(9.3) \quad \lim_{n \rightarrow \infty} \mathbb{P}_{Q^*} \left( \left| E_0^\psi(F((X_{0 \leq t \leq T}^{(n)})) - \mathbb{E}(F((B_{0 \leq t \leq T}))) \right| \geq \epsilon \right) = 0.$$

where  $B_t$  is a  $d$ -dimensional Brownian motion with covariance  $\sigma^2 \text{Id}$ , and where  $Q^*$  is the invariant measure for the processes viewed from the particle

$$Q^* = \frac{\sum_{j \sim 0} W_{0,j} \psi(0) \psi(j)}{\mathbb{E}_{\nu_V^W}(\sum_{j \sim 0} W_{0,j} \psi(0) \psi(j))} \cdot \nu_V^W(d\beta).$$

It is clear, since  $Q^*$  and  $\nu_V^W$  are equivalent probability distribution that (9.3) is also true when  $\mathbb{P}_{Q^*}$  is replaced by  $\mathbb{P}_{\nu_V^W}$ . This implies an annealed functional central limit theorem for the process  $(X_n)$  under the annealed law  $\mathbb{E}_{\nu_V^W}(\tilde{P}_0^\psi(\cdot))$  :

$$(9.4) \quad \lim_{n \rightarrow \infty} \left| \mathbb{E}_{\nu_V^W} \left( E_0^\psi(F((X_{0 \leq t \leq T}^{(n)})) - \mathbb{E}(F((B_{0 \leq t \leq T}))) \right) \right| = 0.$$

Let  $\Upsilon_t^{(n)} := \frac{1}{\sqrt{n}}(X \circ \theta_{\xi_0})_{[nt]}$ . Denote  $d^\circ$  the Skorohod metric on  $D([0, \infty), \mathbb{R}^d)$ , the space of càdlag functions  $f : [0, \infty) \rightarrow \mathbb{R}^d$ . As

$$|X_t^{(n)} - \Upsilon_t^{(n)}| = \frac{1}{\sqrt{n}} |X_{[nt]} - X_{[nt+\xi_0]}| \leq \frac{|\xi_0|}{\sqrt{n}} \xrightarrow{n \rightarrow \infty} 0,$$

we have

$$(9.5) \quad d^\circ(X^{(n)}, \Upsilon^{(n)}) \rightarrow 0.$$

Recall that  $F$  is bounded Lipschitz function for the Skorohod topology, therefore,

$$|F(X_t^{(n)}) - F(\Upsilon_t^{(n)})| \rightarrow 0$$

and (9.4) is valid for  $X^{(n)}$  replaced by  $\Upsilon^{(n)}$ . But  $\Upsilon^{(n)}$  has the same law under  $\tilde{P}_0^\psi$  and  $\tilde{P}_0^{\beta, \gamma, 0}$ . This implies the functional central limit theorem (9.4), for the annealed law  $\mathbb{E}_{\nu_V^W}(\tilde{P}_0^{\beta, \gamma, 0}(\cdot))$  in place of  $\mathbb{E}_{\nu_V^W}(\tilde{P}_0^\psi(\cdot))$  starting from 0. By Theorem 1, the annealed law  $\mathbb{E}_{\nu_V^W}(\tilde{P}_0^{\beta, \gamma, 0}(\cdot))$  is that of the discrete time VRJP.

The proof is exactly the same for the ERRW, one just needs to replace the law  $\nu_V^W(d\beta)$  by the law  $\tilde{\nu}_V^a(dW, d\beta)$ .  $\square$

*Proof of Lemma 7.* Let us start by the ERRW case, ii). Consider the sequence of subsets of  $\mathbb{Z}^d$ ,  $V_n = [-n, n]^d$ . Recall that

$$\psi^{(n)}(j) = e^{u^{(n)}(\delta_n, j)},$$

when  $j \in V_n$ . Consider the point  $y_n = (-n, 0, \dots, 0)$ , so that  $y_n$  is at the boundary of the set,  $y_n \sim \delta_n$ . By Lemma 7 of [8] (which is the ERRW's counterpart of Proposition 7, Section 3.2), we have for  $a > 16$ ,

$$(9.6) \quad \mathbb{E}_{\tilde{\nu}_V^a}((\cosh(u(\delta_n, y_n)))^8) \leq 2,$$

(Indeed, the proof does not depend on the graph structure, nor on the choice of the rooting).

From, [8], Theorem 4, there exists  $0 < \tilde{\lambda}_2 < \infty$  such that if  $a > \tilde{\lambda}_2$ , then for all  $i, j$  in  $V_n$ ,

$$(9.7) \quad \mathbb{E}_{\tilde{\nu}_V^a} \left( \left( (\cosh(u^{(n)}(\delta_n, i)) - u^{(n)}(\delta_n, j)) \right)^8 \right) \leq 2.$$



Remark that in [8], the rooting of the field is at 0 and the graph is the restriction of the graph  $\mathbb{Z}^d$  to  $V_n$ . But an attentive reading of the proof shows that the result is also valid for the graph  $\mathcal{G}_n = (V_n \cup \{\delta_n\}, E_n)$  and rooting  $\delta_n$  as well. Indeed, the estimate is based on the protected Ward's estimates, Lemma 4, which remain valid for diamonds inside the set  $V_n$ , and on the estimate on effective conductances, Proposition 3, which is in fact an estimate inside a "diamond". Remark that the estimate (9.7) is also valid when  $i$  or  $j$  is at the boundary of the set  $V_n$  (in fact the proof is written in the case where the diamond  $R_{i,j}$  is inside the set  $V_n$ , which is the case when  $j = y_n$  and  $i \in \mathbb{Z}^d$  fixed for  $n$  large enough). Specified to  $j = y_n$  and  $i \in \mathbb{Z}^d$  fixed, it gives for  $n$  large enough

$$(9.8) \quad \mathbb{E}_{\tilde{\nu}_V^a} \left( (\cosh(u(\delta_n, i) - u^{(n)}(\delta_n, y_n)))^8 \right) \leq 2.$$

By Cauchy-Schwartz inequality, and by (9.6) and (9.8), we get that

$$\mathbb{E}_{\tilde{\nu}_V^a} \left( (\psi^{(n)}(i))^{\pm 4} \right) \leq \mathbb{E}_{\tilde{\nu}_V^a} \left( e^{\pm 8u^{(n)}(\delta_n, y_n)} \right)^{\frac{1}{2}} \mathbb{E}_{\tilde{\nu}_V^a} \left( e^{\pm 8(u^{(n)}(\delta_n, i) - u^{(n)}(\delta_n, y_n))} \right)^{\frac{1}{2}} \leq C_{\pm}$$

for some constant  $C_{\pm} > 0$  independent of  $n$ . From this we deduce by Fatou's lemma for all  $i, j$  in  $\mathbb{Z}^d$ ,  $i \sim j$ ,

$$\mathbb{E}_{\tilde{\nu}_V^a} \left( ((W_{i,j}\psi(i)\psi(j))^{\pm 1}) \right) \leq \mathbb{E}_{\tilde{\nu}_V^a} \left( (W_{i,j})^{\pm 2} \right)^{\frac{1}{2}} \mathbb{E}_{\tilde{\nu}_V^a} \left( (\psi(0))^{\pm 4} \right)^{\frac{1}{2}} < \infty,$$

for  $a$  large enough.

The proof is very similar in the VRJP case, and uses Theorem 1 of [10]. As previously, the estimate is valid in the case we are interested in, that is for the graph  $\mathcal{G}_n$ , rooted at  $\delta_n$ , and for  $x \in \mathbb{Z}^d$ ,  $y = y_n$  for  $n$  large enough. □

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